Reading for This Lecture

- C&T Chapter 16
Solving the Deterministic Equivalent

Recall the deterministic equivalent (DE) version of the standard two-stage stochastic program.

\[
\begin{align*}
\text{minimize} & \quad c^\top x + p_1 q^\top y_1 + p_2 q^\top y_2 + \cdots + p_s q^\top y_s \\
\text{subject to} & \quad Ax = b \\
& \quad T_1 x + Wy_1 = h_1 \\
& \quad T_2 x + Wy_2 = h_2 \\
& \quad \vdots \ \\
& \quad T_S x + Wy_s = h_s \\
& \quad x \in X, \quad y_1 \in Y, \quad y_2 \in Y, \quad \ldots, \quad y_s \in Y
\end{align*}
\]

Note the block angular structure. How do we take advantage of this?
Bender’s Decomposition

• *Bender’s Decomposition* is a technique for solving LPs with this kind of block angular structure.

• Note that if we fix the first-stage variables ($x$), then the LP decomposes neatly into $|S|$ smaller LPs, one for each scenario.

• Furthermore, these LPs are all identical except for the right-hand side.
Rewriting

- As before, let us rewrite the DE LP as

\[
\text{minimize} \quad c^\top x + \sum_{s \in S} P_s(x)
\]

subject to

\[
Ax = b \quad x \in X
\]

where

\[
P_s(x) = \min_{y \in Y} \{ p_s(q^\top y) \mid Wy = h_s - T_s x \}\
\]
General Solution Approach

• We have already seen that $P_k(x) = v(h_k - T_k x)$ and thus is a convex function (in fact, it is piecewise linear).

• We will linearize the objective function by building up an approximation to it using linear inequalities.

• Essentially, we approximate the LP $P_s(x)$ associated with scenario $s \in S$ by

minimize $z_s$

subject to

$z_s \geq (u^j_s)^\top T_s (x^j - x) + P_s(x^j) \forall j \in J,$

where $J$ indexes a collection of first-stage solutions and $u^j_s$ is an optimal solution to the dual of the LP $P_s(x^j)$ when it exists.

• We will also have to make sure to eliminate any first-stage solution for which there is no feasible recourse in scenario $s$. 
Optimality Cuts

• Let’s consider the LP associated with scenario $s \in S$

\[ P_s(x) = \min_{y \in Y} \{ p_s(q^\top y) \mid W y = h_s - T_s x \} \]

• By LP duality, we have

\[ P_s(x) = \max \{ u_s^\top (h_s - T_s x) \mid W^\top u_s \leq p_s q \} \]

• Let $\hat{x}$ be such that $P_s(\hat{x})$ is feasible and let $\hat{u}_s$ be an optimal dual solution.
Optimality Cuts (cont.)

• Then by LP duality, we have

\[ P_s(x) \geq \hat{u}_s^T (h_s - T_s x) \]

• Furthermore, since

\[ P_s(\hat{x}) = \hat{u}_s^T (h_s - T_s \hat{x}) \]

we have

\[ P_s(x) \geq \hat{u}_s^T T_s (\hat{x} - x) + P_s(\hat{x}) \]
Feasibility Cuts

• Suppose we uncover a first-stage solution $\hat{x}$ for which $P_s(\hat{x})$ is infeasible?

• In this case, we can obtain a direction $\hat{u}_s$ of unboundedness for the dual of the LP $P_s(\hat{x})$.

• For such a direction, we have

$$\hat{u}_s^T(h_s - T_s\hat{x}) > 0$$

and

$$W^T\hat{u}_s \leq p_s q$$

• So we have that $P_s(x)$ will be infeasible for any $x$ such that $\hat{u}_s^T(h_s - T_sx) > 0$.

• Since we are only interested in first-stage solutions with feasible recourse, the inequality

$$\hat{u}_s^T T_s x \geq \hat{u}_s^T h_s$$

must be satisfied by all first-stage solutions.
Reformulation with Benders Cuts

Conceptually, we can reformulate the DE LP as the following LP:

\[
\begin{align*}
\text{minimize} & \quad c^\top x + \sum_{s \in S} z_s \\
\text{subject to} & \quad Ax = b \\
& \quad x \in X \\
& \quad z_s \geq (u_s^j)^\top T_s(x^j - x) + P_s(x^j) \quad \forall j \in J, s \in S,
\end{align*}
\]

- This is an exact reformulation if \( J \) indexes the set of all all first-stage solutions.

- In practice, we maintain a set \( \bar{J} = \{ x^0, \ldots, x^k \} \) of the solutions generated in the first \( k \) iterations, as we see next.
Initializing the Algorithm

• We will start by solving the initial master LP with $\bar{J} = \emptyset$:

  minimize

  \[ c^\top x \]

  subject to

  \[ Ax = b \]

  \[ x \in X \]

  to obtain $x^0$.

• We will then solve $P_s(x^0)$ for each $s \in S$.

• This will give us an upper bound

  \[ c^\top x + \sum_{s \in S} P_s(x^0) \]

  on the optimal value of the stochastic program.
Iterating

- In iteration $k$, we solve $P_s(x^{k-1})$ to obtain either an optimality cut or a feasibility cut (with associated dual solution $\hat{u}_s^{k-1}$).

- We then solve

\[
\begin{align*}
\text{minimize} & \quad c^\top x + \sum_{s \in S} z_s \\
\text{subject to} & \quad Ax = b \\
& \quad x \in X \\
& \quad z_s \geq (u_s^j)^\top T_s(x^j - x) + P_s(x^j) \quad \forall j \in \{0, \ldots, k-1\}, s \in S,
\end{align*}
\]

- This is a relaxation of the original problem, so we get a lower bound and a new first-stage solution $x^1$ (which yields a new upper bound).

- This procedure is iterated until the upper and lower bounds are equal (or at least are “close enough” together).

- This is called the \textit{L-shaped method}. 