Reading for This Lecture

- C&T Chapter 16
Mathematical Programming and Uncertainty

• Many of the models we have seen up until now have involved some degree of uncertainty.
  – Markowitz mean-variance model for portfolio optimization
  – Binomial lattice model for option pricing
  – Similarity model for constructing an index fund

• How did we take account of the inherent randomness in each of these cases?
Modeling Uncertainty

• In all of the previous cases, the stochasticity was handled essentially by creating a deterministic mathematical program that implicitly captures some of the randomness.

• A more direct approach is to create a modeling framework that allows stochasticity to be modeled explicitly.

• There are a number of such frameworks, but we will consider two of them.
  – Stochastic Programming
  – Robust Optimization

• The type of stochasticity addressed in both of these frameworks is uncertainty in the input parameters.

• The ability to model this type of uncertainty is enough to capture the richness of many real-world problems.
A Random Linear Optimization Problem

minimize

\[ x_1 + x_2 \]

subject to

\[ \omega_1 x_1 + x_2 \geq 7 \]
\[ \omega_2 x_1 + x_2 \geq 4 \]
\[ x_1 \geq 0 \]
\[ x_2 \geq 0 \]

\[ \omega_1 \sim U[1, 4] \]
\[ \omega_2 \sim U[1/3, 1] \]
Worth 1000 Words?
What To Do?

• How do we solve this problem?

• What do we mean by solving this problem?

• Suppose it is possible to make a decision after the observation of the random vector \( \omega \)?

• This could be called a “wait-and-see” approach.

• Can we solve the problem then?

• I sure hope so—it’s just a simple deterministic linear optimization problem!
Here and Now

• Generally, “wait-and-see” is not an appropriate model of how things work.

• We generally need to decide on a course of action \textit{before} knowing the outcome of randomness.

• In order for the problem to make sense in this case, we need to decide what to do about not knowing $\omega_1$ and $\omega_2$.

• Three suggestions
  
  – Guess at uncertainty
  – Probabilistic Constraints
  – Penalize Shortfall
First Approach: Guess Away!

- Often, this is what’s done in practice for lack of anything better.
- We simply guess reasonable values for $\omega_1, \omega_2$
- What should we guess?
- Three (obvious) ideas
  - **Unbiased**: Choose mean values
  - **Pessimistic**: Choose worst case values
  - **Optimistic**: Choose best case values
- The choice depends on our level of risk aversion.
Guess: Unbiased

\[ \hat{\omega} \equiv \mathbb{E}(\omega) = (5/2, 3/2) \]

minimize

\[ x_1 + x_2 \]

subject to

\[ \frac{5}{2} x_1 + x_2 \geq 7 \]
\[ \frac{3}{2} x_1 + x_2 \geq 4 \]
\[ x_1, x_2 \geq 0 \]

• \( \text{OPT} = 50/11 \)
• \( (\hat{x}_1, \hat{x}_2) = (18/11, 32/11) \)
Guess: Pessimistic

\( \hat{\omega} = (1, 1/3) \)

minimize

\[ x_1 + x_2 \]

subject to

\[ 1x_1 + x_2 \geq 7 \]
\[ 1/3x_1 + x_2 \geq 4 \]
\[ x_1, x_2 \geq 0 \]

- \( \text{OPT} = 7 \)
- \( (\hat{x}_1, \hat{x}_2) = (0, 7) \)
\[ \hat{\omega} = (4, 1) \]

minimize

\[ x_1 + x_2 \]

subject to

\[ 4x_1 + x_2 \geq 7 \]

\[ x_1 + x_2 \geq 4 \]

\[ x_1, x_2 \geq 0 \]

- \( \text{OPT} = 4 \)
- \( (\hat{x}_1, \hat{x}_2) = (4, 0) \)
Pros and Cons

• Pros
  – All of these approaches are easy
  – We just solve a deterministic problem of the same size as the original random problem.
  – Only “rough” information about the randomness $\omega$ is needed.

• Cons
  – Only takes into account one possible outcome of future uncertainty.
  – There might even be an outcome for which the chosen “solution” is infeasible.
Second Approach: Chance Constrained

Another (perhaps more reasonable) approach is to enforce that the probability of a constraint being satisfied is sufficiently large.

Let’s add the constraints

\[ P\{\omega_1 x_1 + x_2 \geq 7\} \geq \alpha_1 \]
\[ P\{\omega_2 x_1 + x_2 \geq 4\} \geq \alpha_2 \]

Or maybe the constraint

\[ P\{\omega_1 x_1 + x_2 \geq 7, \omega_2 x_1 + x_2 \geq 4\} \geq \alpha \]
Chance Constraints

• Note that for $\alpha_1, \alpha_2, \alpha = 1$, this is equivalent to a normal (deterministic) problem.

• **Question**: How do we solve probabilistically constrained problems?

• **Answer**: It’s extremely difficult

• We will put this method aside for now.
Third Approach: Penalize Shortfall

• We accept infeasibility, but penalize the expected violation.

• Notation:
  – $x^+ \equiv \max(0, z)$: The positive part of $z$.
  – $x^- \equiv \max(0, -z)$: The negative part of $z$.

• Then, for the constraint $\omega_1 x_1 + x_2 \geq 7$, the shortfall is $(\omega_1 x_1 + x_2 - 7)^-$. 

• For each constraint, assign (unit) shortfall costs $q_1, q_2$.

• Optimization problem becomes...

$$\min_{x \in \mathbb{R}^2_+} \left\{ x_1 + x_2 + q_1 \mathbb{E}_{\omega_1} [(\omega_1 x_1 + x_2 - 7)^-] + q_2 \mathbb{E}_{\omega_2} [(\omega_2 x_1 + x_2 - 4)^-] \right\}$$
Yikes!

- The function we are trying to optimize looks ugly.
- However, it is convex.
- In fact, it is not too hard to see that the problem is equivalent to the following:

\[
\min_{x \in \mathbb{R}^2_+} \left\{ x_1 + x_2 + \mathbb{E}_\omega \left[ \min_{y \in \mathbb{R}^2_+} \left\{ q_1 y_1 + q_2 y_2 : \begin{array}{l}
\omega_1 x_1 + x_2 + y_1 \geq 7 \\
\omega_2 x_1 + x_2 + y_2 \geq 4
\end{array} \right\} \right] \right\}
\]
Recourse Function

Let’s write the problem in terms of only the original variables:

\[
\min_{x \in \mathbb{R}_{+}^2} \{ x_1 + x_2 + Q(x_1, x_2) \}
\]

where

\[
Q(x_1, x_2) = \mathbb{E}_\omega \left[ \min_{y \in \mathbb{R}_{+}^2} \left\{ q_1 y_1 + q_2 y_2 : \begin{array}{l} y_1 \geq 7 - \omega_1 x_1 - x_2 \\ y_2 \geq 4 - \omega_2 x_1 - x_2 \end{array} \right\} \right]
\]

- \(Q(x_1, x_2)\) is called the *recourse function*.
- For a given decision \(x_1, x_2\), what do we do (recourse)?
- In this case, the answer is simply to penalize the shortfall.
- \(y_1, y_2\) will be exactly the shortfall in constraints 1 and 2.
Decisions, Stages, and Recourse

Instead of penalizing shortfall, we might be able to take “corrective action,” i.e., recourse!

Consider a planning problem with two periods. The following sequence of events occurs.

1. We make a decision now (first-period decision)
2. A random event occurs ("stuff" happens)
3. We make a second period decision that attempts to repair the havoc wrought by the random event. (recourse)
Example: Farmer Fred’s Magic Beans

• Farmer Fred can grow Wheat, Corn, or Beans on his 500 acres.

• Farmer Fred requires 200 tons of wheat and 240 tons of corn to feed his cattle
  – These can be grown on his land or bought from a wholesaler.
  – Any production in excess of these amounts can be sold for $170/ton (wheat) and $150/ton (corn)
  – Any shortfall must be bought from the wholesaler at a cost of $238/ton (wheat) and $210/ton (corn).

• Farmer Fred can also grow beans
  – Beans sell at $36/ton for the first 6000 tons
  – Due to economic quotas on bean production, beans in excess of 6000 tons can only be sold at $10/ton
The Data

There are 500 acres available for planting

<table>
<thead>
<tr>
<th></th>
<th>Wheat</th>
<th>Corn</th>
<th>Beans</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yield (Tons/acre)</td>
<td>2.5</td>
<td>3</td>
<td>20</td>
</tr>
<tr>
<td>Planting Cost ($/acre)</td>
<td>150</td>
<td>230</td>
<td>260</td>
</tr>
<tr>
<td>Selling Price</td>
<td>170</td>
<td>150</td>
<td>36 (≤ 6000T)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>10 (&gt;6000T)</td>
</tr>
<tr>
<td>Purchase Price</td>
<td>238</td>
<td>210</td>
<td>N/A</td>
</tr>
<tr>
<td>Minimum Requirement</td>
<td>200</td>
<td>240</td>
<td>N/A</td>
</tr>
</tbody>
</table>
LP Formulation: Decision Variables

- $x_{W,C,B}$: Acres of Wheat, Corn, Beans Planted
- $w_{W,C,B}$: Tons of Wheat, Corn, Beans sold (at favorable price).
- $e_B$: Tons of beans sold at lower price
- $y_{W,C}$: Tons of Wheat, Corn purchased.

Notes:
- Farmer Fred has recourse.
- After he observes the weather, he can decide how much of each crop to sell or purchase!
**LP Formulation: Objective and Constraints**

maximize

\[-150x_W - 230x_C - 260x_B - 238y_W + 170w_W - 210y_C + 150y_C + 36w_B + 10e_B\]

subject to

\[
\begin{align*}
  x_W + x_C + x_B & \leq 500 \\
  2.5x_W + y_W - w_W & = 200 \\
  3x_C + y_C - w_C & = 240 \\
  20x_B - w_B - e_B & = 0 \\
  w_B & \leq 6000 \\
  \end{align*}
\]

\[
\begin{align*}
  x_W, x_C, x_B, y_W, y_C, e_B, w_W, w_C, w_B & \geq 0 
\end{align*}
\]
## Solution with (expected) yields

<table>
<thead>
<tr>
<th></th>
<th>Wheat</th>
<th>Corn</th>
<th>Beans</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Plant (acres)</strong></td>
<td>120</td>
<td>80</td>
<td>300</td>
</tr>
<tr>
<td><strong>Production</strong></td>
<td>300</td>
<td>240</td>
<td>6000</td>
</tr>
<tr>
<td><strong>Sales</strong></td>
<td>100</td>
<td>0</td>
<td>6000</td>
</tr>
<tr>
<td><strong>Purchase</strong></td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Profit:** $118,600
Planting Intuition

The LP solution corresponds to Farmer Fred's intuition.

- Plant the land necessary to grow up to his quota limit of beans.
- Plant land necessary to meet his requirements for wheat and corn
- Plant remaining land with wheat – sell excess.
It’s the Weather, Stupid!

- Farmer Fred knows enough to know that his yields aren’t always precisely $Y = (2.5, 3, 20)$.

- He decides to run two more scenarios:
  - Good weather: $1.2Y$
  - Bad weather: $0.8Y$
Formulation: Good weather

maximize

\[-150x_W - 230x_C - 260x_B - 238y_W + 170w_W - 210y_C + 150y_C + 36w_B + 10e_B\]

subject to

\[
\begin{align*}
    x_W + x_C + x_B & \leq 500 \\
    3x_W + y_W - w_W & = 200 \\
    3.6x_C + y_C - w_C & = 240 \\
    24x_B - w_B - e_B & = 0 \\
    w_B & \leq 6000 \\
    x_W, x_C, x_B, y_W, y_C, e_B, w_W, w_C, w_B & \geq 0
\end{align*}
\]
**Solution: Good Weather**

<table>
<thead>
<tr>
<th></th>
<th>Wheat</th>
<th>Corn</th>
<th>Beans</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plant (acres)</td>
<td>183.33</td>
<td>66.67</td>
<td>250</td>
</tr>
<tr>
<td>Production</td>
<td>550</td>
<td>240</td>
<td>6000</td>
</tr>
<tr>
<td>Sales</td>
<td>350</td>
<td>0</td>
<td>6000</td>
</tr>
<tr>
<td>Purchase</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Profit:** $167,667
Formulation: Bad Weather

maximize

\[-150x_W - 230x_C - 260x_B - 238y_W + 170w_W - 210y_C + 150y_C + 36w_B + 10e_B\]

subject to

\[
\begin{align*}
x_W + x_C + x_B & \leq 500 \\
2x_W + y_W - w_W & = 200 \\
2.4x_C + y_C - w_C & = 240 \\
16x_B - w_B - e_B & = 0 \\
w_B & \leq 6000 \\
x_W, x_C, x_B, y_W, y_C, e_B, w_W, w_C, w_B & \geq 0
\end{align*}
\]
## Solution: Bad Weather

<table>
<thead>
<tr>
<th></th>
<th>Wheat</th>
<th>Corn</th>
<th>Beans</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plant (acres)</td>
<td>100</td>
<td>25</td>
<td>375</td>
</tr>
<tr>
<td>Production</td>
<td>200</td>
<td>60</td>
<td>6000</td>
</tr>
<tr>
<td>Sales</td>
<td>0</td>
<td>0</td>
<td>6000</td>
</tr>
<tr>
<td>Purchase</td>
<td>0</td>
<td>180</td>
<td>0</td>
</tr>
</tbody>
</table>

Profit: $59,950
What To Do?

- Obviously the answer is quite dependent on the weather and the respective yields.
- Without knowing the weather, he can’t determine the proper acreage of beans to plant.
- It’s impossible to make a perfect decision, since planting decisions must be made now, but purchase and sales decisions can be made later.
Answer: Maximize Expected Profit

- Assume that the three scenarios occur with equal probability.
- Attach a scenario subscript $s = 1, 2, 3$ to each of the purchase and sale variables (1=Good, 2=Average, 3=Bad).
  - $w_{C2}$: Tons of corn sold at favorable price in scenario 2
  - $e_{B3}$: Tons of beans sold at unfavorable price in scenario 3.
Expected Profit

An expression for Farmer Fred’s Expected Profit is the following:

\[
150x_W - 230x_C - 260x_B \\
+ \frac{1}{3}(-238y_{W1} + 170w_{W1} - 210y_{C1} + 150y_{C1} + 36w_{B1} + 10e_{B1}) \\
+ \frac{1}{3}(-238y_{W2} + 170w_{W2} - 210y_{C2} + 150y_{C2} + 36w_{B2} + 10e_{B2}) \\
+ \frac{1}{3}(-238y_{W3} + 170w_{W3} - 210y_{C3} + 150y_{C3} + 36w_{B3} + 10e_{B3})
\]
Expected Value Problem: Constraints

\[
\begin{align*}
x_W + x_C + x_B & \leq 500 \\
3x_W + y_{W1} - w_{W1} & = 200 \\
2.5x_W + y_{W2} - w_{W2} & = 200 \\
2x_W + y_{W3} - w_{W3} & = 200 \\
3.6x_C + y_{C1} - w_{C1} & = 240 \\
3x_C + y_{C2} - w_{C2} & = 240 \\
2.4x_C + y_{C3} - w_{C3} & = 240 \\
24x_B - w_{B1} - e_{B1} & = 0 \\
20x_B - w_{B2} - e_{B2} & = 0 \\
16x_B - w_{B3} - e_{B3} & = 0 \\
w_{B1}, w_{B2}, w_{B3} & \leq 6000 \\
\text{All vars} & \geq 0
\end{align*}
\]
## Expected Value Problem: Solution

<table>
<thead>
<tr>
<th>s</th>
<th>Plant (acres)</th>
<th>Wheat</th>
<th>Corn</th>
<th>Beans</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Production</td>
<td>510</td>
<td>288</td>
<td>6000</td>
</tr>
<tr>
<td>1</td>
<td>Sales</td>
<td>310</td>
<td>48</td>
<td>6000</td>
</tr>
<tr>
<td>1</td>
<td>Purchase</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>Production</td>
<td>425</td>
<td>240</td>
<td>5000</td>
</tr>
<tr>
<td>2</td>
<td>Sales</td>
<td>225</td>
<td>0</td>
<td>5000</td>
</tr>
<tr>
<td>2</td>
<td>Purchase</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>Production</td>
<td>340</td>
<td>192</td>
<td>4000</td>
</tr>
<tr>
<td>3</td>
<td>Sales</td>
<td>140</td>
<td>0</td>
<td>4000</td>
</tr>
<tr>
<td>3</td>
<td>Purchase</td>
<td>0</td>
<td>48</td>
<td>0</td>
</tr>
</tbody>
</table>

**Profit:** $108,390
Examinining the Solution

- Best solution allocates land for beans to always avoid having to sell them at the unfavorable price.
- Corn is planted so that the requirement is met in the average scenario.
- The remaining land is allocated to wheat.
- Again, it is impossible to find a solution that is ideal under all circumstances.
- Decisions in stochastic models are balanced, or hedged against the various scenarios.
Fortune Telling

• Suppose Farmer Fred could *with certainty* tell whether or not the upcoming growing season was going to have good weather, average weather, or bad weather.
  
  – His bursitis was acting up
  – Consulting the Farmer’s Almanac
  – Hire a fortune teller

• One question here is how much Farmer Fred would be willing to pay for this “perfect” information.

• In real-life problems, how much is it “worth” to invest in better (or perfect) forecasting technology?
What’s it worth?

• With perfect information, Farmer Fred’s would plant (wheat, corn, beans).
  – Good yield: (183.33, 66.67, 250), Profit: $167,667
  – Average yield: (120, 80, 300), Profit: $118,600
  – Bad yield: (100, 25, 375), Profit: $59,950

• Assuming each of these scenarios occurs with probability 1/3, his long run average profit would be

\[
\frac{1}{3}(167667) + \frac{1}{3}(118600) + \frac{1}{3}(59950) = 115406
\]

• With his (optimal) “here-and-now” decision of (170, 80, 250), he would make a long run profit of 108390

• This difference (115406-108390) is the expected value of perfect information (EVPI)