Reading for This Lecture

- C&T Chapter 15
The Mortgage Market

- Mortgages represent the largest single sector of the U.S. debt market, surpassing even the federal government.

- Many financial instruments have therefore been created to provide credit to this market.

- The primary way this has been accomplished since the 1970s is the bundling together of individual mortgages into capital market instruments called *mortgage-backed securities* (MBSs).

- The principal and interest from the mortgages in the pool backing an MBS are passed through to investors in some fashion.

- By selling MBSs, banks can realize their fees up front and lay off their risk to the market.
Pass-through MBSs

• Initially, MBSs were simply packaged using a pass-through structure.

• Each investor received a pro rata share of principal and interest payments for mortgages in the pool.

• The problem with this approach is that the cash flows are very unpredictable due to *pre-payment risk*.

• Mortgage payers prepay for a variety of reasons, but for fixed-rate mortgages, this is usually associated with a drop in interest rates.

• This may force an unplanned reinvestment at a lower interest rate.
Collateralized Mortgage Obligations

- A *collateralized mortgage obligation* is a more sophisticated MBS that rearranges cash flows to make them more predictable.

- There are many ways of doing this, but here we focus on the creation of *consecutive tranches*.

- The basic idea is to package the cash flows into bonds with different maturities.

- Principal payments are funneled to investors in each tranche consecutively until the obligation is repaid.
Simple Two-Tranche Model

- Suppose we have an MBS consisting of $100 million in mortgage loans.
- In a two-tranche model, we might divide the pool into two $50 million tranches.
- Initially, investors in both tranches receive interest payments, but all principal payments are funneled to the investors in the first tranche (the "fast-pay tranche") until it is repaid.
- After the fast-pay tranche is repaid, remaining principal payments go to the second tranche.
- By restructuring in this way, the fast-pay tranche reaches maturity much earlier than the "slow-pay tranche."
- A byproduct of the restructuring is that the risk of default is much lower for the fast-pay tranche.
- This means that the interest rate paid on the fast-pay tranche can be reduced, resulting in additional profit.
A Model of Consecutive Tranches

- Early payments are more likely to be fully funded than later ones.
- Hence, fast tranches get a higher credit rating than slower ones and can be sold at lower interest rates.
- Overall, the interest that has to be paid to buyers of the tranches is lower than the interest paid by the mortgage holders.
- Hence, the bank issuing the restructured tranches earns money.
- A bond with payback $p_t$ of principal at time $t$ ($t = 1, \ldots, T$) is priced with respect to its weighted average life (WAL)

$$WAL = \frac{\sum_{t=1}^{T} tp_t}{\sum_{t=1}^{T} p_t}.$$ 

- A bond with a WAL of $n$ years will be priced like a treasury bond with a duration of $n$ years plus a spread (extra interest) which depends on the credit rating.
Credit Ratings and Spreads

- The spot rates of future coupon payments and the spreads for different credit ratings can be looked up in a table.

- For example, the table could look as follows, where the spreads under the credit ratings (AAA etc.) are given in basis points, i.e., in \(1/100\) of 1%.

<table>
<thead>
<tr>
<th>duration</th>
<th>spot rate</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.74%</td>
<td>85</td>
<td>100</td>
<td>115</td>
<td>130</td>
<td>165</td>
<td>220</td>
<td>345</td>
</tr>
<tr>
<td>2</td>
<td>4.89%</td>
<td>90</td>
<td>105</td>
<td>125</td>
<td>140</td>
<td>190</td>
<td>275</td>
<td>425</td>
</tr>
<tr>
<td>3</td>
<td>5.05%</td>
<td>95</td>
<td>110</td>
<td>135</td>
<td>150</td>
<td>210</td>
<td>335</td>
<td>500</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
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<td>...</td>
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<td>...</td>
</tr>
</tbody>
</table>
Notation

• $Q_0$ is the amount of principal to be repaid.
• $T$ is the horizon over which the principal is to be repaid.
• $I_t$ is the interest to paid in year $t$.
• $A_t$ is the scheduled amortization payment in year $t$.
• $q_t$ is the pre-payment rate in year $t$.
• $R_t$ is the pre-payment amount in year $t$.
• $P_t$ is the total payment in year $t$ (including pre-payment).
• $Q_t$ is the amount of principal outstanding at the end of year $t$.
• $r \times 100\%$ is the compound yearly interest rate.
Example

Let’s take $Q_0 = 100$, $r = 0.1$, $T = 10$ $q_1 = .01\%$. If the principal is to be repaid in equal installments, then the scheduled amortization payment in year 1 is

$$A_1 = \frac{Q_0 r}{(1 + r)^T - 1} = 6.27.$$ 

So we have for year 1:

- Interest: $I_1 = rQ_0 = 10$.
- Scheduled amortization: $A_1 = Q_0 r / [(1 + r)^T - 1] = 6.27$.
- Prepayment: $R_1 = q_1(Q_0 - 6.27) = 0.937$.
- Total principal pay down: $P_1 = R_1 + A_1 = 6.27 + 0.937 = 7.207$.
- Principal left after year 1: $Q_1 = Q_0 - P_1 = 92.793$. 
Generalizing

In general, we have a given scenario $q_1, \ldots, q_T$ of prepayment rates in years 1, 2, \ldots, $T$. In year $t$, we have

- Interest: $I_t = rQ_{k-1}$.
- Scheduled amortization: $A_t = Q_{k-1}r / [(1 + r)^T - 1]$.
- Prepayment: $R_t = q_t(Q_{k-1} - A_t)$.
- Total principal pay down: $P_t = R_t + A_t$.
- Principal left after year 1: $Q_t = Q_{k-1} - P_t$.

One can thus recursively compute the corresponding $(I_t, P_t, Q_t)$ ($t = 1, \ldots, T$).
Packaging

• The model we have presented is a simplification of the real problem.

• In real CMOs, the *pay-back time* of the principal is also variable, rather than just the *amount* of principal paid back.

• Nevertheless, this model is a good approximation to the real one and leads to very similar results.

• Once the payouts $P_t$ are known, the question is how to optimally package them into consecutive tranches

$$ (P_1, \ldots, P_{T_1}), (P_{T_1+1}, \ldots, P_{T_2}), \ldots, (P_{\ldots}, \ldots, P_T). $$
Candidate Tranches

• Let us refer to the candidate tranche \((P_j, \ldots, P_t)\) as \((j, t)\).
• Associated with the candidate tranche \((j, t)\) is its buffer

\[
B_{jt} = \frac{\sum_{k=t+1}^{T} P_k}{\sum_{k=1}^{T} P_k},
\]

• The buffer is the proportion of principal left after the tranche expires.
• Each tranche also has its own WAL

\[
WAL_{jt} = \frac{\sum_{k=j}^{t} kP_k}{\sum_{k=j}^{t} P_k}.
\]

• Note that this is the WAL of a bond that has no repayment of principal for the first \(j - 1\) years, but interest (coupons) is still paid during this time.
Prepayment Scenarios

• In order to achieve a high quality ranking, a tranche must be able to sustain higher than expected default rates without compromising payments to the tranche holders.

• The default rate is determined by the scenario of prepayment rates $q_1, \ldots, q_T$.

• Regulatory bodies require that several prescribed scenarios be tested.

• For example, the Public Securities Association (PSA) industry standard benchmark is $q_1 = 0.01$, $q_2 = 0.03$, $q_3 = 0.05$, and $q_t = 0.06$ for $t \geq 4$. 
Tranche Credit Ratings

- For a tranche to be given a certain credit rating, it must satisfy

\[ B_{jt} \geq W A L_{jt} \cdot d \cdot L, \]

where \( L \) is the *loss multiple*, specified as follows,

<table>
<thead>
<tr>
<th>Rating</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1.5</td>
<td>0</td>
</tr>
</tbody>
</table>

- Hence, the earlier tranches naturally receive higher credit ratings.
Present Value of a Tranche

- From \((B_{jt}, W_{jt})\) and the above table one can thus compute the credit rating for each candidate tranche \((j, t)\).
- This rating implies a coupon rate \(c_{jt}\) that can be read off the earlier table of spot rates and spreads.
- Using the coupon rates \(c_{jt}\), the net present value \(Z_{jt}\) of tranche \((j, t)\) can be computed:
  - In period \(k\), a payment of \(c_{jt}\) times the remaining principal on the tranche is paid (as interest), and if \(k \in [j, t]\), then the principal payment \(P_k\) is made.
  - The result is a total payment of \(C_k\).
  - Then the present value is \(T_{jt} = \sum_{k=1}^{t} C_k / (1 + r_k)^k\).
A Dynamic Programming Formulation

• To maximize earnings, the issuer now wants to structure the CMO into $K$ sequential tranches so as to minimize the net present value of total payments to bond-holders.

• The stages will be the number of tranches and the states will be the years $1, \ldots, T$.

• We set the value function to be

\[
v(k, t) = \begin{align*}
&\text{The minimum present value of total payments to} \\
&\text{bondholders in years } 1 \text{ through } t \\
&\text{when the CMO has } k \text{ tranches up to year } t.
\end{align*}
\]

• Then

\[
v(1, t) = T_{1,t} \\
v(k, t) = \min_{j=k-1,\ldots,t-1} \{v(k - 1, j) + T_{j+1,t}\}, k \leq t.
\]
Finding the Optimal CMO Structure

- Using this recurrence, we compute $v(k, t)$ for $k = 1, \ldots, K$ and $t = 1, \ldots, T$.
- The optimal net present value of future payments to bondholders is then $\min_{k=1,\ldots,K}(k, T)$. 