Reading for This Lecture

- C&T Chapter 14
Option Pricing Revisited

- Recall the analysis from Lecture 6 in which we determined the fair price of an option.
- The price model for the underlying asset considered just one time period and two scenarios.
- Here, we extend this single period pricing model to consider an option on an asset whose price evolves over $n$ periods.
- We will assume the option has a strike price of $K$. 
Initial Conditions

• The price of the asset is $S_0$ at time 0.

• As before, the price at time 1 is a random variable $S_1$ over a probability space $(\Omega^1, P^1)$.

• We have a partition of $\Omega^1$ into two disjoint events $\Omega^1_0$ and $\Omega^1_1$ such that $P^1(\Omega^1_0) = 1 - P^1(\Omega^1_1) = p$ for a given parameter $0 < p < 1$.

• Then

$$S_1(\omega) = \begin{cases} uS_0 & \text{if } \omega \in \Omega^1_0, \\ dS_0 & \text{if } \omega \in \Omega^1_1, \end{cases}$$

where parameters $u > 1$ and $d < 1$ are given.
The Binomial Lattice

- The price at time $k$ is a random variable $S_k$ over a probability space $(\Omega^k, P^k)$.
- Suppose we have already observed the price at time $k - 1$ to be $\alpha$.
- Then the price at time $k$ will be such that the asset takes value $u\alpha$ with probability $p$ or $d\alpha$ with probability $1 - p$.
- We can visualize the evolution of the price as a “binomial lattice” in which the lattice points correspond to the possible prices in each period.
Price Scenarios

- After $k$ periods, the price must be $u^j d^{k-j} S_0$ for some $j$.
- This corresponds to $j$ periods in which the price went up and $k - j$ periods in which the price went down.
- Hence, the number of possible price scenarios for the asset in period $k$ is $k + 1$.
- Recall that the probability of an “up move” is $p$ and the probability of a “down move” is $1 - p$.
- Therefore, the probability of any path followed through the lattice that ends with price $u^j d^{k-j} S_0$ is $p^j (1 - p)^{k-j}$.
- The number of such paths is the number of ways of choosing the $j$ “up periods,” i.e., the number of paths is $\binom{k}{j}$.
Distribution of $S_k$

- For the previous slide, $S_k$ follows a standard binomial distribution with parameters $k$ and $p$ on the probability space $(\Omega^k, P^k)$.

- Formally, we assume that $\Omega^k$ is partitioned into disjoint events $\Omega^k_0, \ldots, \Omega^k_k$.

- Then we have

$$P^k(\Omega^k_j) = \binom{k}{j} p^j (1 - p)^{k-j}$$

and

$$S_k(\omega) = u^j d^{k-j} S_0 \quad \forall j \in 0, \ldots, k, \omega \in \Omega^k_j$$
Estimating the Parameters

- Before using the model, we need to choose values of $u$, $d$, and $p$.
- This is done so that the resulting price distribution has the same mean and variance as the asset itself.
- Because the model is multiplicative, it is convenient to use logarithms.
- Let $\mu$ and $\sigma$ be the mean and standard deviation of $\ln\left(\frac{S_n}{S_0}\right)$, which we assume is known.
- Let $\Delta = 1/n$ be the length of time between periods.
- Then the mean and standard deviation of $\ln\left(\frac{S_1}{S_0}\right)$ is $\mu \Delta$ and $\sigma \sqrt{\Delta}$.
- It is easy to compute that the mean and variance of $\ln\left(\frac{S_1}{S_0}\right)$ are $p \ln u + (1 - p) \ln d$ and $p(1 - p)(\ln u - \ln d)^2$. 
Estimating the Parameters (cont.)

- Matching these values, we get two equations

\[ p \ln u + (1 - p) \ln d = \mu \Delta \]
\[ p(1 - p)(\ln u - \ln d)^2 = \sigma^2 \Delta \]

- Note that there are now two equations and three parameters.
- To get a solution, we further simplify by requiring \( d = 1/u \).
- When \( \Delta \) is small, the equations can be solved approximately to yield the values

\[ u = e^{\sigma \sqrt{\Delta}} \]
\[ d = e^{-\sigma \sqrt{\Delta}} \]
\[ p = \frac{1}{2} \left( 1 + \frac{\mu}{\sigma} \sqrt{\Delta} \right) \]
Pricing the Option

- Once we have the parameters, the option price can be determined using a dynamic programming approach.
- The stages are the time periods and the states correspond to the possible prices of the asset.
- The value function is defined such that $v(j, k)$ is the value of an option held in period $k$ given that the current asset price is $u^j d^{k-j} S_0$.
- Although this is not really an optimization problem, we can still use the DP approach.
The Recursion

• We use a backwards recursion from the final period to compute the value $v(0, 0)$.

• In the last period, there is no stochasticity and the value in state $j$ for a call option is simply

$$v(N, j) = \max\{u^j d^{N-j} S_0 - K, 0\}$$

and a for a put option

$$v(N, j) = \max\{K - u^j d^{N-j} S_0, 0\}$$

• Using the technique from Lecture 6, we can compute $v(k, j)$ by knowing $v(k + 1, j)$ and $v(k + 1, j + 1)$.

• As before, this is done using the risk-neutral probabilities.

$$p_u = \frac{R - d}{u - d} \quad \text{and} \quad p_d = \frac{u - R}{u - d},$$

where $R = 1 + r$ and $r$ is the one-period risk-free rate of return.
The Recursion (cont.)

- For a European option, we get

\[ v(k, j) = \frac{1}{R}(p_u v(k + 1, j + 1) + p_d v(k + 1, j)) \]

- For an American call option, we get

\[ v(k, j) = \max\{ \frac{1}{R}(p_u v(k + 1, j + 1) + p_d v(k + 1, j)), u^j d^{k-j} S_0 - K \} \]

- For an American put option, we get

\[ v(k, j) = \max\{ \frac{1}{R}(p_u v(k + 1, j + 1) + p_d v(k + 1, j)), K - u^j d^{k-j} S_0 \} \]
A Model for Optimal Exercise Decisions

• Let us now look at a more general price evolution model.

• We now let the price of a given asset in period $k$ be a continuous random variable given by

$$S_k = S_{k-1} + X_k,$$

where $X_k$ is a random variable with mean $\mu$ and distribution function $F$.

• The parameter $\mu$ can be interpreted as the mean return over one period.

• We assume that $X_j$ and $X_k$ are i.i.d. for any two periods $j$ and $k$.

• We would like to know the optimal exercise policy for an American call option on this asset with strike price $K$. 
A Dynamic Programming Model

- We will develop a continuous DP model for this situation.
- In such a DP, the number of states in each stage can be infinite.
- As before, the stages correspond to \( N + 1 \) time periods and the states correspond to the price of the asset.
- Now, however, the price distribution is continuous.
- The value function \( v(k, S) \) will be the maximum expected profit given that the asset has price \( S \) and the option expires in \( k \) days.
- Note that this means stage 0 corresponds to period \( N \) and stage \( N \) corresponds to period 0.
A Dynamic Programming Model

- The decision set for each state is to either exercise the option or not.
- The immediate benefit from exercising is $S - K$.
- Otherwise, we are choosing to wait at least one more period.
- Given this formulation, the recursion is

$$v(k, S) = \max\{S - K, \int v(k - 1, S + x)dF(x)\}.$$ 

with the boundary condition $v(0, S) = \max\{S - K, 0\}$.
- There is no closed form for $v(k, S)$.
- Using DP, we can obtain a solution numerically.
The Optimal Policy

Theorem 1. The optimal policy for an American call option has the following form. There are nondecreasing numbers $s_1 \leq s_2 \leq \cdots \leq s_k \leq \cdots \leq s_N$ such that if the current stock price is $S$ and there are $k$ days until expiration, one should exercise the option if and only if $S \geq s_k$. 