Reading for This Lecture

• Nemhauser and Wolsey Sections II.1.1-II.1.3, II.1.6
• Wolsey Chapter 8
• CCZ Chapters 5 and 6
• “Valid Inequalities for Mixed Integer Linear Programs,” G. Cornuejols.
• “Generating Disjunctive Cuts for Mixed Integer Programs,” M. Perregaard.
Generating Cutting Planes: Two Basic Viewpoints

• There are a number of different points of view from which one can derive the standard methods used to generate cutting planes for general MILPs.

• As we have seen before, there is an algebraic point of view and a geometric point of view.

• **Algebraic:**
  – Take combinations of the known valid inequalities.
  – Use rounding to produce stronger ones.

• **Geometric:**
  – Use a disjunction (as in branching) to generate several disjoint polyhedra whose union contains $S$.
  – Generate inequalities valid for the convex hull of this union.

• Although these seem like very different approaches, they turn out to be very closely related.
Generating Valid Inequalities: Algebraic Viewpoint

• Consider the polyhedron \( Q = \{ x \in \mathbb{R}^n \mid Ax \leq b \} \).

• Valid inequalities for \( Q \) can be obtained by taking non-negative linear combinations of the rows of \((A, b)\).

• Except for one pathological case\(^1\), all valid inequalities for \( Q \) are either equivalent to or dominated by an inequality of the form

\[
uAx \leq ub, \ u \in \mathbb{R}_+^m.\]

• We are taking combinations of inequalities existing in the description, so any such inequalities will be redundant for \( Q \) itself.

• Nevertheless, such redundant inequalities can be strengthened by a simple procedure when \( Q \) is the LP relaxation of an MILP.

\(^1\)The pathological case occurs when one or more variables have no explicit upper bound and both the primal and dual problems are infeasible.
Generating Valid Inequalities for \( \text{conv}(S) \)

As usual, we consider the MILP

\[
    z_{IP} = \max \{ c^\top x \mid x \in S \}, \quad \text{(MILP)}
\]

where

\[
    \mathcal{P} = \{ x \in \mathbb{R}^n \mid Ax \leq b \} \quad \text{(FEAS-LP)}
\]

\[
    S = \mathcal{P} \cap (\mathbb{Z}_+^p \times \mathbb{R}_+^{n-p}) \quad \text{(FEAS-MIP)}
\]

- All inequalities valid for \( \mathcal{P} \) are also valid for \( \text{conv}(S) \), but they are not cutting planes.
- We can do better.
- We need the following simple principle: if \( a \leq b \) and \( a \) is an integer, then \( a \leq \lfloor b \rfloor \).
- Believe it or not, this simple fact is all we need to generate all valid inequalities for \( \text{conv}(S) \)!
Simple Example

• Suppose $4x_1 + 2x_2 \leq 3$ is an inequality in the formulation $\mathcal{P}$ for a given MILP.

• Dividing through by 2, we get that $2x_1 + x_2 \leq 3/2$ is also valid for $\mathcal{P}$.

• Using the rounding principle, we can easily derive that $2x_1 + x_2 \leq 1$ is valid for $\text{conv}(\mathcal{S})$. 
Back to the Matching Problem

Recall again the matching problem.

\[
\begin{align*}
\min & \quad \sum_{e=\{i,j\} \in E} c_e x_e \\
\text{s.t.} & \quad \sum_{\{j\mid \{i,j\} \in E\}} x_{ij} = 1, \quad \forall i \in N \\
x_e & \in \{0, 1\}, \quad \forall e = \{i,j\} \in E.
\end{align*}
\]
Generating the Odd Cut Inequalities

• Recall that each odd cutset induces a possible valid inequality.

\[ \sum_{e \in \delta(S)} x_e \geq 1, \; S \subset N, \; |S| \text{ odd.} \]

• Let’s derive these another way.
  
  – Consider an odd set of nodes \( U \).
  – Sum the (relaxed) constraints \( \sum_{\{i,j\}\in E} x_{ij} \leq 1 \) for \( i \in U \).
  – This results in the inequality \( 2 \sum_{e \in E(U)} x_e + \sum_{e \in \delta(U)} x_e \leq |U| \).
  – Dividing through by 2, we obtain \( \sum_{e \in E(U)} x_e + \frac{1}{2} \sum_{e \in \delta(u)} x_e \leq \frac{1}{2} |U| \).
  – We can drop the second term of the sum to obtain

\[ \sum_{e \in E(U)} x_e \leq \frac{1}{2} |U|. \]

  – What’s the last step?
Chvátal Inequalities

• Suppose we can find a $u \in \mathbb{R}_+^m$ such that $\pi = uA$ is integer and $\pi_0 = ub \notin \mathbb{Z}$.

• In this case, we have $\pi^T x \in \mathbb{Z}$ for all $x \in S$, and so $\pi^T x \leq \lfloor \pi_0 \rfloor$ for all $x \in S$.

• In other words, $(\pi, \lfloor \pi_0 \rfloor)$ is both a valid inequality and a split disjunction.

• In other words, it is a split disjunction for which

$$\{x \in P \mid \pi^T x \geq \lfloor \pi_0 \rfloor + 1\} = \emptyset$$

(1)

• Such an inequality is called a Chvátal inequality.

• Note that we have not used the non-negativity constraints in deriving this inequality.
Chvátal-Gomory Inequalities

• Taking into account that we assume \( S \) is contained in the non-negative orthant, we can derive a Chvátal inequality from any \( u \in \mathbb{R}^n_+ \), as follows.

• First, we have that \((uA, ub)\) is valid for \( P \).

• Since the variables are non-negative, we have that \( uA_C x_C \geq 0 \), so

\[
\sum_{i=1}^{p} (uA_i)x_i \leq ub \forall x \in P
\]

• Again because the variables are non-negative, we have that

\[
\sum_{i=1}^{p} \lfloor uA_i \rfloor x_i \leq ub \forall x \in P
\]

• Finally, we have that

\[
\sum_{i=1}^{p} \lfloor uA_i \rfloor x_i \leq \lfloor ub \rfloor \forall x \in S,
\]

which is a Chvátal inequality known as a Chvátal-Gomory Inequality.
Chvátal-Gomory Inequalities

- Chvátal-Gomory (C-G) inequalities can also be derived in another way.
- We explicitly add the non-negativity constraints to the formulation along with the other linear constraints with associated multipliers $v \in \mathbb{R}^n_+$. 
- We cannot round the coefficients to make them integral, so we require $\pi$ integral.

$$
\pi_i = uA_i - v_i \in \mathbb{Z} \text{ for } 1 \leq i \leq p
$$

$$
\pi_i = uA_i - v_i = 0 \text{ for } p + 1 \leq i \leq n.
$$

- $v_i$ will be non-negative as long as we have

$$
v_i \geq uA_i - \lfloor uA_i \rfloor \quad \text{for } 1 \leq i \leq p
$$

$$
v_i = uA_i \geq 0 \quad \text{for } p + 1 \leq i \leq n
$$

- Taking $v_i = uA_i - \lfloor uA_i \rfloor$ for $1 \leq i \leq p$, we then obtain that

$$
\sum_{0 \leq i \leq p} \pi_i x_i = \sum_{0 \leq i \leq p} [uA_i] x_i \leq [ub] = \pi_0 \quad \text{(C-G)}
$$

is a C-G inequality for all $u \in \mathbb{R}^m_+$ such that $uA_C \geq 0$. 
The Chvátal-Gomory Procedure

1. Choose a weight vector \( u \in \mathbb{R}^m_+ \) such that \( uA_C \geq 0 \).
2. Obtain the valid inequality \( \sum_{0 \leq i \leq p} (uA_i)x_i \leq ub \).
3. Round the coefficients down to obtain \( \sum_{0 \leq i \leq p} (\lfloor uA_i \rfloor)x_i \leq ub \).
4. Finally, round the right-hand side down to obtain the valid inequality

\[
\sum_{0 \leq i \leq p} (\lfloor uA_i \rfloor)x_i \leq \lfloor ub \rfloor
\]

- This procedure is called the Chvátal-Gomory rounding procedure, or simply the C-G procedure.
- Surprisingly, for pure ILPs \( p = n \), any inequality valid for \( \text{conv}(S) \) can be produced by a finite number of applications of this procedure!
- Note that this procedure is recursive and requires exploiting inequalities derived in previous rounds to get new inequalities.
- This is not true for the general mixed case.
Assessing the Procedure

- Although it is theoretically possible to generate any valid inequality using the C-G procedure, this is not true in practice.

- The two biggest challenges are numerical errors and slow convergence.

- The inequalities produced may be very weak—we may not even obtain a supporting hyperplane.

- This is because the rounding only “pushes” the inequality until it meets some point in $\mathbb{Z}^n$, which may or may not even be in $S$.

- The coefficients of the generated inequality must be relatively prime to ensure the generated hyperplane even includes an integer point!

**Proposition 1.** Let $S = \{ x \in \mathbb{Z}^n \mid \sum_{j \in N} a_j x_j \leq b \}$, where $a_j \in \mathbb{Z}$ for $j \in N$, and let $k = \gcd\{a_1, \ldots, a_n\}$. Then $\text{conv}(S) = \{ x \in \mathbb{R}^n \mid \sum_{j \in N} \left( a_j/k \right) x_j \leq \lfloor b/k \rfloor \}$. 
Another Simple Example

- Consider \( S = \{ x \in \mathbb{Z}^n \mid 2x_1 + 4x_2 \leq 3 \} \).

- The gcd of the coefficients of this inequality is 2 and the right-hand side is odd.

- It is clear that no integer point satisfies \( 2x_1 + 4x_2 = 3 \), so the corresponding inequality cannot be a facet of \( \text{conv}(S) \).

- Thus, if we applied rounding to an inequality \( 2x_1 + 4x_2 \leq 7/2 \), the hyperplane \( 2x_1 + 4x_2 = 3 \) would only include \( \frac{1}{2} \) - integral points.

- If the gcd of the coefficients divides the right-hand side after rounding, then the associated hyperplane would contain integer points.
Gomory Inequalities

- Let's consider $T$, the set of solutions to a pure ILP with one equation:

$$ T = \left\{ x \in \mathbb{Z}_+^n \mid \sum_{j=1}^n a_j x_j = a_0 \right\} $$

- For each $j$, let $f_j = a_j - \lfloor a_j \rfloor$ and let $f_0 = a_0 - \lfloor a_0 \rfloor$. Then equivalently

$$ T = \left\{ x \in \mathbb{Z}_+^n \mid \sum_{j=1}^n f_j x_j = f_0 + \lfloor a_0 \rfloor - \sum_{j=1}^n \lfloor a_j \rfloor x_j \right\} $$

- Since $\sum_{j=1}^n f_j x_j \geq 0$ and $f_0 < 1$, then $\lfloor a_0 \rfloor \geq \sum_{j=1}^n \lfloor a_j \rfloor x_j$ so

$$ \sum_{j=1}^n f_j x_j \geq f_0 $$

is a valid inequality for $S$ called a Gomory inequality.
Gomory Cuts from the Tableau

• Gomory cutting planes can also be derived directly from the tableau while solving an LP relaxation.

• We assume for now that $A$ and $b$ are integral so that the slack variables also have integer values implicitly (this is wlog if $P$ is rational).

• Consider the set

$$\{(x, s) \in \mathbb{Z}_{+}^{n+m} \mid Ax + Is = b\}$$

in which the LP relaxation of an ILP is put in standard form.

• The tableau corresponding to basis matrix $B$ is

$$B^{-1}Ax + B^{-1}s = B^{-1}b$$

• Each row of this tableau corresponds to a weighted combination of the original constraints.

• The weight vectors are the rows of $B^{-1}$. 
Gomory Cuts from the Tableau (cont.)

- A row of the tableau is obtained by combining the equations in the standard representation with weight vector $\lambda = B_j^{-1}$ to obtain

$$\sum_{j=1}^{n} (\lambda A_j) x_j + \sum_{i=1}^{m} \lambda_i s_i = \lambda b,$$

where $A_j$ is the $j^{th}$ column of $A$ and $\lambda$ is a row of $B^{-1}$.

- Applying the previous procedure, we can obtain the valid inequality

$$\sum_{j=1}^{n} (\lambda A_j - [\lambda A_j]) x_j + \sum_{i=1}^{m} (\lambda_i - [\lambda_i]) s_i \geq \lambda b - [\lambda b]. \quad \text{(GF)}$$

- We then typically substitute out the slack variables by using the equation $s = b - Ax$ to obtain this cut in the original space.

$$\sum_{j=1}^{n} \left( [\lambda A_j] - \sum_{i=1}^{m} [\lambda_i] a_{ij} \right) x_j \leq [\lambda b] - \sum_{i=1}^{m} [\lambda_i] b_i.$$

- This Gomory cut is equivalent to the C-G inequality with weights $u_i = \lambda_i - [\lambda_i]$, as we show next.
Gomory Versus C-G

• To begin, we let $u_i = \lambda_i - \lfloor \lambda_i \rfloor$, so that

$$uAx = \lambda Ax - \lfloor \lambda \rfloor Ax \leq \lambda b - \lfloor \lambda \rfloor b = ub$$

• Rounding then yields

$$\sum_{j=1}^{n} \left( \lfloor \lambda A_j \rfloor - \sum_{i=1}^{m} \lfloor \lambda_i \rfloor a_{ij} \right) x_j \leq \lfloor \lambda b \rfloor - \sum_{i=1}^{m} \lfloor \lambda_i \rfloor b_i,$$

• Substituting $s = b - Ax$, we get

$$\sum_{j=1}^{n} \lfloor \lambda A_j \rfloor x_j + \sum_{i=1}^{m} \lfloor \lambda_i \rfloor s_i \leq \lfloor \lambda b \rfloor. \quad (2)$$

• Finally, subtracting (2) from

$$\lambda Ax + \lambda s = \lambda b,$$

we obtain the Gomory fractional cut (GF).
Strength of Gomory Cuts from the Tableau

- Consider a row of the tableau in which the value of the basic variable is not an integer.

- Applying the procedure from the last slide, the resulting inequality will only involve nonbasic variables and will be of the form

\[ \sum_{j \in NB} f_j x_j \geq f_0 \]

where \( 0 \leq f_j < 1 \) and \( 0 < f_0 < 1 \).

- The left-hand side of this cut has value zero with respect to the solution to the current LP relaxation.

- We can conclude that the generated inequality will be violated by the current solution to the LP relaxation.
Example: Gomory Cuts

Consider the polyhedron $\mathcal{P}$ described by the constraints

$$4x_1 + x_2 \leq 28 \quad (3)$$
$$x_1 + 4x_2 \leq 27 \quad (4)$$
$$x_1 - x_2 \leq 1 \quad (5)$$
$$x_1, x_2 \geq 0 \quad (6)$$

Graphically, it can be easily determined that the facet-inducing valid inequalities describing $\text{conv}(S = \mathcal{P} \cap \mathbb{Z}^2)$ are

$$x_1 + 2x_2 \leq 15 \quad (7)$$
$$x_1 - x_2 \leq 1 \quad (8)$$
$$x_1 \leq 5 \quad (9)$$
$$x_2 \leq 6 \quad (10)$$
$$x_1 \geq 0 \quad (11)$$
$$x_2 \geq 0 \quad (12)$$
Example: Gomory Cuts (cont.)

Figure 1: Convex hull of $S$
Example: Gomory Cuts (cont.)

Consider the optimal tableau of the LP relaxation of the ILP

$$\max\{2x_1 + 5x_2 \mid x \in S\},$$

shown in Table 1.

<table>
<thead>
<tr>
<th>Basic var.</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>-2/30</td>
<td>8/30</td>
<td>0</td>
<td>16/3</td>
</tr>
<tr>
<td>$s_3$</td>
<td>0</td>
<td>0</td>
<td>-1/3</td>
<td>1/3</td>
<td>1</td>
<td>2/3</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>8/30</td>
<td>-2/30</td>
<td>0</td>
<td>17/3</td>
</tr>
</tbody>
</table>

Table 1: Optimal tableau of the LP relaxation

The associated optimal solution to the LP relaxation is also shown in Figure 1.
Example: Gomory Cuts (cont.)

The Gomory cut from the first row is

$$\frac{28}{30}s_1 + \frac{8}{30}s_2 \geq \frac{1}{3},$$

In terms of $x_1$ and $x_2$, we have

$$4x_1 + 2x_2 \leq 33,$$

(G-C1)

Note this inequality can be trivially strengthened by dividing by 2.
Example: Gomory Cuts (cont.)

The Gomory cut from the second row is

\[ \frac{2}{3}s_1 + \frac{1}{3}s_2 \geq \frac{2}{3}, \]

In terms of \( x_1 \) and \( x_2 \), we have

\[ 3x_1 + 2x_2 \leq 27, \quad \text{(G-C2)} \]
Example: Gomory Cuts (cont.)

The Gomory cut from the third row is

\[ \frac{8}{30}s_1 + \frac{28}{30}s_2 \geq \frac{2}{3}, \]

In terms of \( x_1 \) and \( x_2 \), we have

\[ x_1 + 2x_2 \leq 16, \]  \hspace{1cm} (G-C3)
Example: Gomory Cuts (cont.)

This picture shows the effect of adding all Gomory cuts in the first round.
Applying the Procedure Recursively

• This procedure can be applied recursively by adding the generated inequalities to the formulation and performing the same steps again.

• Any inequality that can be obtained by recursive application of the C-G procedure (or is dominated by such an inequality) is a C-G inequality.

• For pure ILPs, all valid inequalities are C-G inequalities.

**Theorem 1.** Let \((\pi, \pi_0) \in \mathbb{Z}^{n+1}\) be a valid inequality for \(S = \{x \in \mathbb{Z}_+^n \mid Ax \leq b\} \neq \emptyset\). Then \((\pi, \pi_0)\) is a C-G inequality for \(S\).

• In the next few slides, we will make these ideas more precise.
Elementary Closure

- The elementary closure, or C-G closure, of a polyhedron \( \mathcal{P} \subseteq \mathbb{R}^n_+ \) is the intersection of half-spaces defined by C-G inequalities, e.g.,

\[
e(\mathcal{P}) = \{ x \in \mathcal{P} \mid \pi^\top x \leq \pi_0, \pi_j = \lfloor u a_j \rfloor \text{ for } 1 \leq j \leq p, \\
\pi_j = 0 \text{ for } p + 1 \leq j \leq n, \pi_0 = \lfloor u b \rfloor, u \in \mathbb{R}^m_+ \}\]

- Although it is not obvious, one can show that the elementary closure is a polyhedron.

- Optimizing over this polyhedron is difficult (NP-hard) in general.

- For a general polyhedron \( \mathcal{P} \), not necessarily contained in the non-negative orthant, we can similarly define the Chvátal closure.

\[
\mathcal{P}^{CH} = \{ x \in \mathcal{P} \mid \pi^\top x \leq \pi_0, \pi = u A, \pi_0 = \lfloor u b \rfloor, u A_I \in \mathbb{Z}^p, u A_O = 0 \}\]


**Rank of C-G Inequalities**

- The *rank k C-G closure* $P^k$ of $P$ is defined recursively as follows.
  - The rank 1 closure of $P$ is $P^1 = e(P)$.
  - The rank $k$ closure $P^k = e(P^{k-1})$ is the elementary closure of the $P^{k-1}$.
  - An inequality is rank $k$ if it is valid for the rank $k$ closure $P^k$ and not for $P^{k-1}$.

- The **C-G rank** of $P$ is the maximum rank of any facet-defining inequality of $\text{conv}(S)$.

- We can define a similar notion of rank with respect to the Chvátal closure.
A Finite Cutting Plane Procedure

• Under mild assumptions on the algorithm used to solve the LP, this yields a general algorithm for solving (pure) ILPs.

• The details are contained in Section 5.2.5 of CCZ.
Determining the C-G Rank

• By solving an LP, it can be determined whether a given inequality has maximum rank 1.

Proposition 2. If \((\pi, \pi_0) \in e(P)\), then \(\pi_0 \geq \lfloor \pi^L_P \rfloor\), where \(\pi^L_P = \max_{x \in P} \pi^\top x\)

• Alternatively, if \(\pi \in \mathbb{Z}^n\), the inequality \((\pi, \lfloor \pi^L_P \rfloor)\) is rank 1.

• Further, any valid inequality \((\pi, \pi_0)\) for which \(\pi_0 < \lfloor \pi^L_P \rfloor\) has rank at least 2.

• This tells us that the effectiveness of the C-G procedure is strongly tied to the strength of our original formulation.

• In general it is difficult to determine the rank of any inequality that is not rank 1.
Example: C-G Rank

• Let’s consider the C-G rank of the inequality

\[ x_1 + 2x_2 \leq 15, \]

which is facet-defining for \( \text{conv}(S) \) in our example.

• We have

\[ \max_{x \in P} x_1 + 2x_2 = \frac{49}{3}. \quad (13) \]

• Since \( \lfloor \frac{49}{3} \rfloor = 16 \), we conclude that this is not a rank 1 cut.

• Note that the dual solution to the LP (13) gives us weights with which to combine the original inequalities to get a C-G cut.

• This is the strongest possible C-G cut of rank 1 with those coefficients.
Bounding The C-G Rank of a Polyhedron

• For most classes of MILPs, the rank of the associated polyhedron is an unbounded function of the dimension.

• **Example:**
  
  – $\mathcal{P} = \{ x \in \mathbb{R}_+^n \mid x_i + x_j \leq 1 \text{ for } i, j \in N, i \neq j \}$ and $S = \mathcal{P}^n \cap \mathbb{Z}^n$
  
  – $\text{conv}(S) = \{ x \in \mathbb{R}_+^n \mid \sum_{j \in N} x_j \leq 1 \}$.
  
  – $\text{rank}(\mathcal{P}) = O(\log n)$.

• For a family of polyhedra with bounded rank, there is a certificate for the validity of any given inequality.

• This leads to a certificate of optimality for the associated optimization problem.

• Hence, it is unlikely that the problem of optimizing over any family of MILPs formulated by polyhedra with bounded rank is in NP-hard\textsuperscript{2}.

• Conversely, for any family of MILPs that is in NP-hard, the associated family of polyhedra is likely to have unbounded rank.

\textsuperscript{2}More on what this means later