The Value Function of a Mixed Integer Linear Program

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Outline

- Consider the MILP

$$\min_{x \in \mathcal{S}} cx, \qquad \text{(P)}$$

$c \in \mathbb{R}^n$, $S = \{x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+ \mid Ax = b\}$ with $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{R}^m$

- The value function of the primal problem (P) is

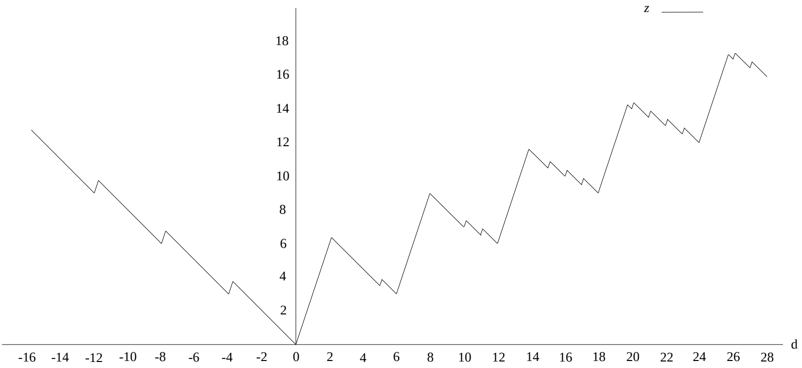
$$z(d) = \min_{x \in \mathcal{S}(d)} cx,$$

where for a given $d \in \mathbb{R}^m$, $\mathcal{S}(d) = \{x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+ \mid Ax = d\}$

- $I = \{1, \ldots, r\}$, $C = \{r+1, \ldots, n\}$, $N = I \cup C$
- $z(d) = \infty$ if $\mathcal{S}(d) = \emptyset$
- $z(0) = 0$

Example

$$\begin{array}{lll} \min & 3x_1 + \frac{7}{2}x_2 + 3x_3 + 6x_4 + 7x_5 + 5x_6 & \\ \text{s.t} & 6x_1 + 5x_2 - 4x_3 + 2x_4 - 7x_5 + x_6 = b \quad \text{and} & \text{(SP)} \\ & x_1, x_2, x_3 \in \mathbb{Z}_+, x_4, x_5, x_6 \in \mathbb{R}_+ & \end{array}$$



Structure of the Value Function

- It is subadditive over $\Omega = \{d \mid \mathcal{S}(d) \neq \emptyset\}$
- It is piecewise polyhedral
- For pure integer case ($C = \emptyset$), it can be obtained by a finite number of limited operations on elements of the RHS:

$$\left.\begin{array}{l}\left.\begin{array}{l}\text{(i) rational multiplication}\\ \text{(ii) nonnegative combination}\\ \text{(iii) rounding}\end{array}\right\}\textit{Chvátal fcns.}\\ \text{(iv) taking the maximum}\end{array}\right\}\textit{Gomory fcns.}$$

Structure of the Value Function

Mixed Case:

- Let the set $\mathscr{E}$ consist of the index sets of dual feasible bases of the linear program

$$\min\{\frac{1}{D}c_C x_C \ : \ \frac{1}{D}A_C x_C = b, x \geq 0\}$$

  where $D \in \mathbb{Z}_+$ such that for any $E \in \mathscr{E}$, $DA_E^{-1}a^j \in \mathbb{Z}^m$ for all $j \in I$

- Jeroslow Formula: There is a Gomory function g such that

$$z(d) = \min_{E \in \mathscr{E}} g(\lfloor d \rfloor_E) + v_E(d - \lfloor d \rfloor_E) \ \ \forall d \in \mathbb{R}^m \ \text{ with } \ \mathcal{S}(d) \neq \emptyset,$$

  where for $E \in \mathscr{E}$, $\lfloor d \rfloor_E = A_E \lfloor A_E^{-1} d \rfloor$ and v_E is the corresponding basic feasible solution

Structure of the Value Function

Maximal Subadditive Extension:

- All the information to obtain the value function is contained in a neighborhood defined by the constraint matrix coefficients.
- Let $M = \{1, \ldots, m\}$ and assume that $A \in \mathbb{Q}^m_+$, $\Omega = \mathbb{R}_+$. Let $q \in \mathbb{Q}^m_+$ be defined as

$$q_i = \max\{a_{ij} \mid j \in N\},$$

and let

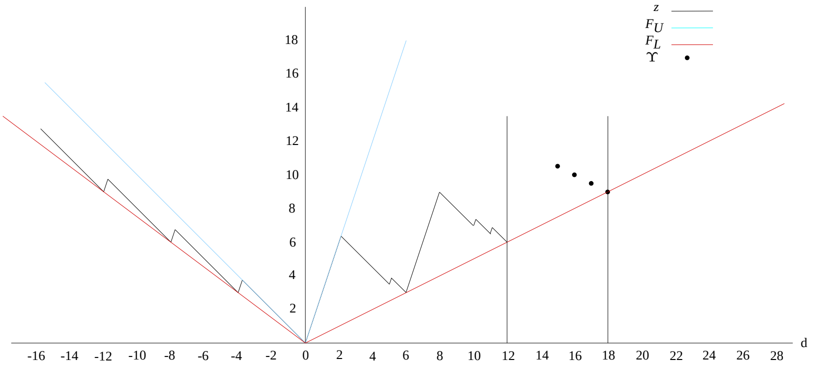
$$F(d) = \begin{cases} z(d) & \text{if } \ d_i \in [0, q_i] i \in M \\ \min\limits_{\mathcal{C} \in \mathcal{C}(d)} \sum\limits_{\rho \in \mathcal{C}} z(\rho) & \text{if } \ d_i \notin [0, q_i] \text{ for some } i \in M \end{cases}, \qquad (1)$$

Then, $z(d) = F(d) \ \ \forall d \in \mathbb{R}^m_+$.

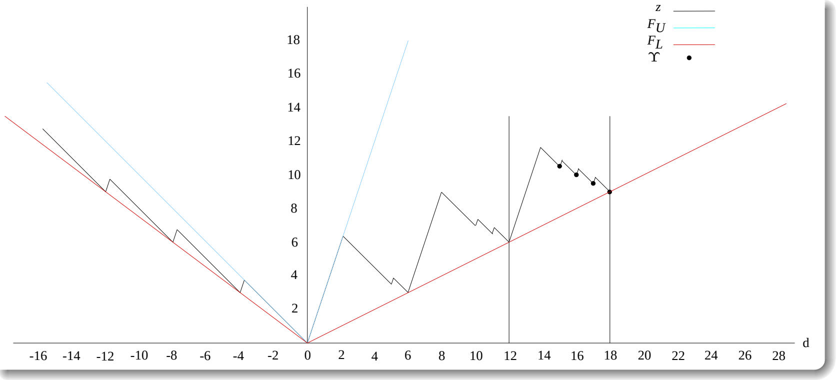
  - $\mathcal{C}(d)$: the set of all finite collections $\{\rho^1, \ldots, \rho^R\}$, $\rho^j \in \mathbb{R}^m_+$ such that $\rho^j \in \times_{i \in M}[0, q_i], j = 1, \ldots, R$ and $\sum_{j=1}^R \rho^j = d$.

Example (cont'd)

Extending the value function of sample MILP from $[0, 12]$ to $[0, 18]$



Example (cont'd)



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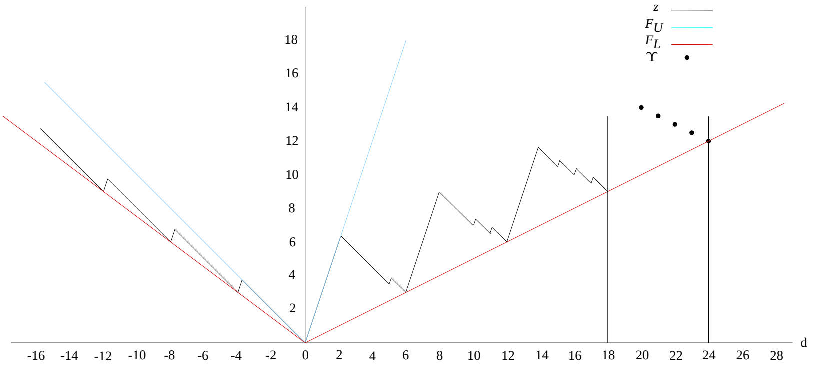
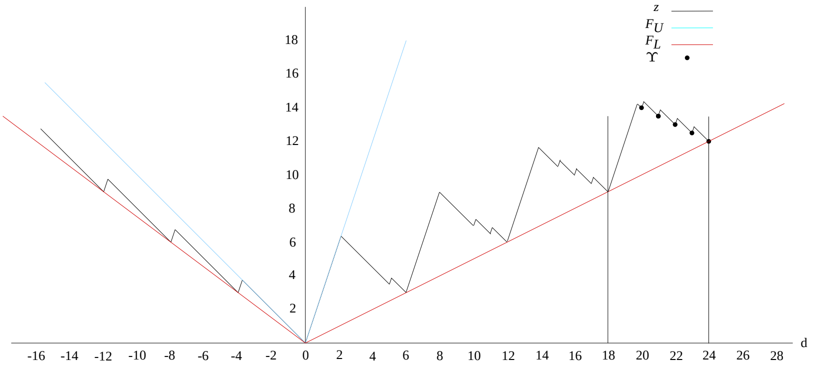


Example (cont'd)



Duality

- A *dual function* $F : \mathbb{R}^m \rightarrow \mathbb{R}$ is a function that bounds the value function

$$F(d) \leq z(d) \quad \forall d \in \mathbb{R}^m$$

- Duality in mathematical programming is the study of such dual functions and the methods for constructing them
- Dual functions are the basis of lower bounding approximations
- A dual function F is called strong with respect to a given right hand side b, if $F(b) = z(b)$
- For a given b, there exists a subadditive, strong dual function
- How?

Constructing Dual Functions

- Explicit construction
  - The Value Function (Jeroslow & Blair -77/79/82/84, Blair -95, Williams -96)
  - Generating Functions - Z transformation (Lasserre -04)
  - Generating Functions - Short Rational Functions, Global Test Sets (Loera et al. [2004])
- Primal Solution Algorithms
  - Cutting Plane Method (Gomory -69, Chvátal -73, Wolsey -81)
  - Branch-and-Bound Method (Wolsey -81)
  - Branch-and-Cut Method
- Relaxations
  - Lagrangian Relaxation (Fisher -81)
  - Quadratic Lagrangian Relaxation (Jeroslow & Blair -79, wolsey -81)
  - Corrected Linear Dual Functions - Linear Represantation (Jeroslow & Blair -78, Wolsey -81, Lasserre -04)
  - Single Row Approximations - Maximal Subadditive Extension

Dual Function from Branch and Bound

- Solve the primal problem with branch and bound.
- T: the set of leaf nodes
- $(v^t, \underline{v}^t, \overline{v}^t)$: the dual solution of LP relaxation solved at node $t \in T$.
- Dual function:

$$F_{BB}(d) = \min_{t \in T}\{v^t d + \underline{v}^t l^t - \overline{v}^t u^t\}$$

- F_{BB} is piecewise linear, concave and strong with respect to b

Dual Function from Branch and Cut

- We need to *generalize* the valid inequalities so that for a given $d \neq b$, T would still be a valid partition
- It is sufficient to update the right hand side of each cut:
  - subadditive representation is known: Ex. Gomory Mixed Integer Cuts

$$\sum_{j \in I} F_\alpha(\lambda A_j) + \sum_{j \in C} \bar{F}_\alpha(\lambda A_j) \geq F_\alpha(\lambda b),$$

where $\lambda \in \mathbb{R}^m_+$, $0 \leq \alpha < 1$

$$F_\alpha(h) = \begin{cases} h - \lfloor h \rfloor & h - \lfloor h \rfloor \leq \alpha \\ h - \left(\lfloor h \rfloor + \frac{\max\{h - \lfloor h \rfloor, 0\} - \alpha}{1 - \alpha} \right) & h - \lfloor h \rfloor > \alpha \end{cases}$$

and

$$\bar{F}_\alpha(h) = \begin{cases} h & h \geq 0 \\ \frac{h}{1-\alpha} & h < 0 \end{cases}$$

  - Right hand side dependency is known: Ex. Cover Inequalities
  - In worst case, variable bounds can be used

Dual Function from Branch and Cut

- Dual function:

$$F_{BC}(d) = \min_{t \in T} \{ v^t d + \underline{v}^t l^t - \overline{v}^t u^t + \sum_{k=1}^{\delta^t} w_k^t F_k^t(\sigma_k^t(d)) \}$$

  - $(v^t, \underline{v}^t, \overline{v}^t, w^t)$: the dual solution of LP relaxation solved at node $t \in T$.
  - δ^t: the number of cuts generated on the path (including the ones generated at t)
  - σ_k^t: recursive dependency of right hand side of cut k at leaf node t on the original right hand side
  - F_k^t: right hand side dependency function of cut k at leaf node t

- F_{BC} is strong with respect to b

Dual Function from Relaxations

- Consider the value functions of each single row relaxation:

$$z_i(q) = \min\{cx \mid a_i x = q, x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+\}\ q \in \mathbb{R}, i \in M \equiv \{1, \ldots, m\}$$

  where a_i is the i^{th} row of A.

- Dual Function:

$$F_{SR}(d) = \max_{i \in M}\{z_i(d_i)\},\ d = (d_1, \ldots, d_m),\ d \in \mathbb{R}^m.$$

- F_{SR} is subadditive but not necessarily strong with respect to a given right hand side

Dual Function from Relaxations

- Let $K \subseteq M$,and define $\phi : [0, q_K] \rightarrow \mathbb{R}$ as

$$\phi(h) = \max_{i \in K}\{z_i(h_i)\} \; \forall h \in [0, q_K].$$

  where q_k is the vector of maximum coefficients of each row in set K
- Let G_K be the maximal subadditive extension of ϕ to $\mathbb{R}_+^{|K|}$
- Dual Function:

$$F_{MS}(d) = \max \left\{ G_K(d_k), \max_{i \in M \setminus K} \{z_i(d_i)\} \right\} \; \forall d \in \mathbb{R}_+^m$$

- F_{MS} is subadditive

Dual Function from Relaxations

- For $K \subseteq M$, $\lambda \in \mathbb{R}_+^{|K|}$, set

$$G_K(h, \omega) = \min\{cx \mid \lambda a_K x = \lambda h, x \in \mathbb{Z}_+^r \times \mathbb{R}_+^{n-r}\} \; \forall h \in \mathbb{R}^{|K|}$$

- Dual Function:

$$F_A(\omega, d) = \max\left\{G_K(d_K, \omega), \max_{i \in M \setminus K}\{z_i(d_i)\}\right\}, \; d \in \mathbb{R}^m$$

- F_A is subadditive

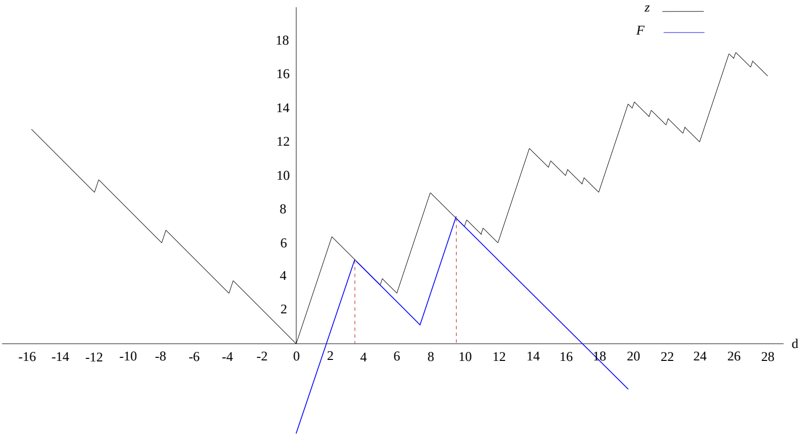
Global Lower Bounding Approximation

- $\mathscr{F}$: the set of the dual functions obtained for each $d \in U \subset \mathbb{R}^m$ where U is some collection of right-hand sides
- Then, we call the dual function F defined as

$$F(d) = \max_{f \in \mathscr{F}} \{f(d)\} \quad \forall d \in \mathbb{R}^m$$

  a global approximation of the value function
- This global approximation can be used to substitute the value function in larger instances

Example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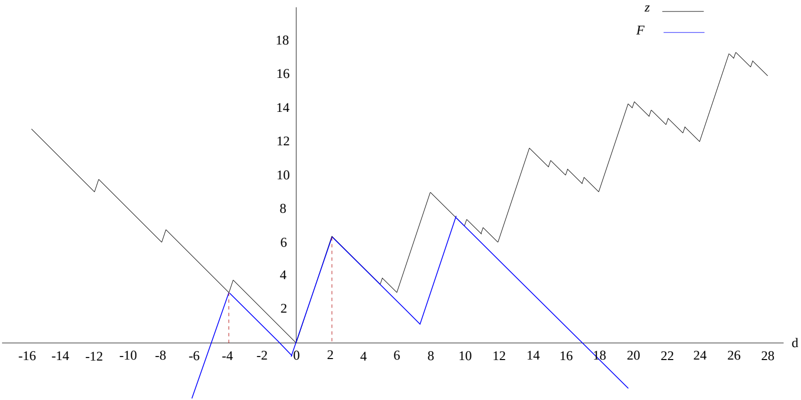
Example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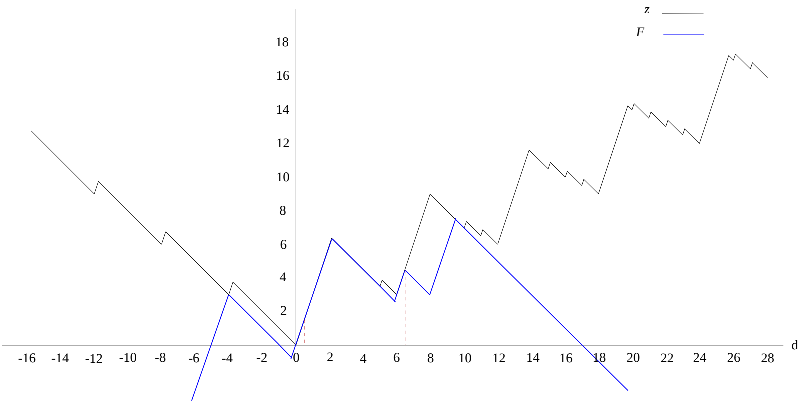
Example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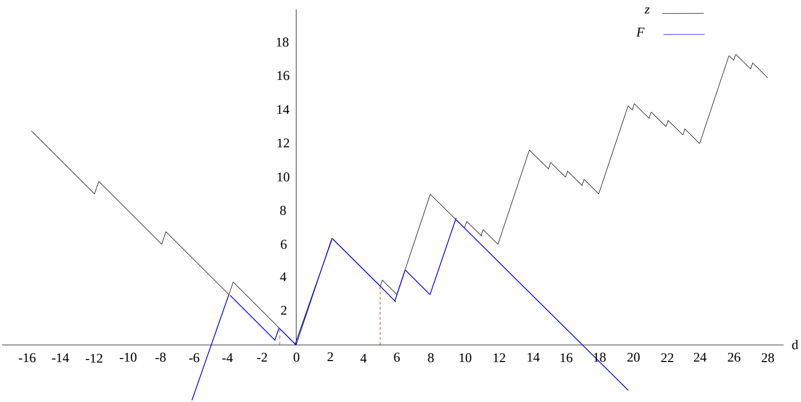
Example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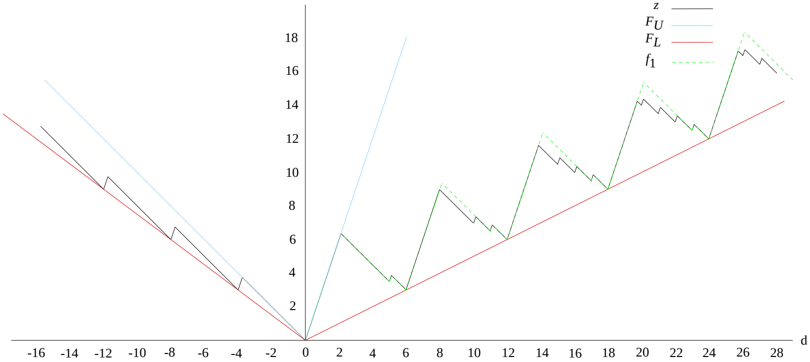
Upper Bounding Approximations

- Just like the lower bounding approximations, we are interested in a function $F : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{\infty\}$ such that

$$F(d) \geq z(d) \quad \forall d \in \mathbb{R}^m$$

- Closely related to feasibility, so it is harder to obtain compared to dual functions
- One possible way is to consider the maximal subadditive extension result:
  - Let $f(d) \geq z(d)$ for all $d \in [0, q]$
  - Then, any extension F of f from $[0, q]$ to $\mathbb{R}^m_+$ is an upper bounding function
  - Any extension rule will do

Example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Upper Bounding Approximations

- Another way is to consider restrictions
- We will only consider fixing variables:
  - Let $K \subset N$, $s \in \mathbb{R}_+^{|K|}$ be given and define the function

$$F(d) = c_K s + z_{N\backslash K}(d - A_K s) \;\; \forall d \in R^m,$$

where

$$\begin{aligned} z_{N\backslash K}(h) = \min \quad & c_{N\backslash K} y \\ s.t \quad & A_{N\backslash K} y = h \\ & y \in \mathbb{Z}_+^{r_2} \times \mathbb{R}_+^{n_2 - r_2}, n_2 = |N\backslash K| \end{aligned}$$

  - Then, $F(d) \geq z(d) \quad \forall d \in \mathbb{R}^m$ if $s \in \mathbb{Z}_+^{r_1} \times \mathbb{R}_+^{n_1 - r_1}, n_1 = |K|$
- For a given right-hand side b, if $s = x_K^*$ where x^* is an optimal solution, then F is strong at b

Upper Bounding Approximations

- A practical one: set $K = I$
- Assume that $\{u \in \mathbb{R}^m \mid uA_C \leq c_C\}$ is not empty and bounded
- Let x^* be an optimal solution to the primal problem with right-hand side set to b
- Then the upper bounding function reduces to

$$F(d) = c_I x_I^* + z_C(d - A_I x_I^*) \quad \forall d \in \mathbb{R}^m$$

  with

$$z_C(h) = \max\{vh \mid v \in V\} \quad \forall h \in R^m$$

  where V is the set of extreme points of dual polytope
- F is piecewise-linear, convex and strong at b.

Global Upper Bounding Approximations

Example