On the Value Function of a Mixed Integer Linear Program

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Outline

1. Introduction

2. Structure of The Value Function
   - Definitions
   - Linear Approximations
   - Properties

3. Constructing the Value Function
   - Subadditive Extension
   - Algorithms
   - General Case

4. Current and Future Work
Motivation

The goal of this work is to study the structure of the value function of a general mixed integer linear program (MILP).

We hope this will lead to methods for approximation useful for

- Sensitivity analysis
- Warm starting
- Multi-level/hierarchical mathematical programming
- Other methods that require dual information

Constructing the value function (or even an approximation to it) is difficult, even in a small neighborhood.

We begin by considering the value functions of single-row relaxations.
We consider the MILP

\[
\min_{x \in S} cx, \quad (P)
\]

\(c \in \mathbb{R}^n, S = \{x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+ | a'x = b\}\) with \(a \in \mathbb{Q}^n, b \in \mathbb{R}\).

The value function of \((P)\) is

\[
z(d) = \min_{x \in S(d)} cx,
\]

where for a given \(d \in \mathbb{R}, S(d) = \{x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+ | a'x = d\}\).

Assumptions: Let \(I = \{1, \ldots, r\}, C = \{r + 1, \ldots, n\}, N = I \cup C\).

- \(z(0) = 0 \implies z : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}\),
- \(N^+ = \{i \in N | a_i > 0\} \neq \emptyset\) and \(N^- = \{i \in N | a_i < 0\} \neq \emptyset\),
- \(r < n\), that is, \(|C| \geq 1 \implies z : \mathbb{R} \to \mathbb{R} \).
Example

\[
\begin{align*}
\min & \quad 3x_1 + \frac{7}{2}x_2 + 3x_3 + 6x_4 + 7x_5 + 5x_6 \\
\text{s.t} & \quad 6x_1 + 5x_2 - 4x_3 + 2x_4 - 7x_5 + x_6 = b \quad \text{and} \\
& \quad x_1, x_2, x_3 \in \mathbb{Z}^+, x_4, x_5, x_6 \in \mathbb{R}^+. \\
\end{align*}
\]
Simple Bounding Functions

- $F_L$: LP Relaxation $\rightarrow$ Lower Bounding function
- $F_U$: Continuous Relaxation $\rightarrow$ Upper Bounding function

\[
F_L(d) = \begin{cases} 
\eta d & \text{if } d > 0, \\
0 & \text{if } d = 0, \\
\zeta d & \text{if } d < 0.
\end{cases}
\]

\[
F_U(d) = \begin{cases} 
\eta^C d & \text{if } d > 0 \\
0 & \text{if } d = 0 \\
\zeta^C d & \text{if } d < 0
\end{cases}
\]

where, setting $C^+ = \{i \in C \mid a_i > 0\}$ and $C^- = \{i \in C \mid a_i < 0\}$,

\[
\eta = \min\left\{\frac{c_i}{a_i} \mid i \in N^+\right\} \quad \text{and} \quad \zeta = \max\left\{\frac{c_i}{a_i} \mid i \in N^-\right\}
\]

\[
\eta^C = \min\left\{\frac{c_i}{a_i} \mid i \in C^+\right\} \quad \text{and} \quad \zeta^C = \max\left\{\frac{c_i}{a_i} \mid i \in C^-\right\}.
\]

- By convention: $C^+ \equiv \emptyset \rightarrow \eta^C = \infty$ and $C^- \equiv \emptyset \rightarrow \zeta^C = -\infty$.
- $F_U \geq z \geq F_L$
Example (cont’d)

\[ \eta = \frac{1}{2}, \zeta = -\frac{3}{4}, \eta^C = 3 \text{ and } \zeta^C = -1: \]

\[ \begin{align*}
\{ \eta = \eta^C \} & \iff \{ z(d) = F_U(d) = F_L(d) \quad \forall d \in \mathbb{R}_+ \} \\
\{ \zeta = \zeta^C \} & \iff \{ z(d) = F_U(d) = F_L(d) \quad \forall d \in \mathbb{R}_- \} 
\end{align*} \]
Consider $d_U^+, d_U^-, d_L^+, d_L^-:$

The relation between $F_U$ and the linear segments of $z$: $\{\eta^C, \zeta^C\}$
Redundant Variables

Let \( T \subseteq C \) be such that

- \( t^+ \in T \) if and only if \( \eta^C < \infty \) and \( \eta^C = \frac{c_t^+}{a_t^+} \) and similarly,
- \( t^- \in T \) if and only if \( \zeta^C > -\infty \) and \( \zeta^C = \frac{c_t^-}{a_t^-} \).

and define

\[
\nu(d) = \min \; c_I x_I + c_T x_T \\
\text{s.t.} \; a_I x_I + a_T x_T = d \\
x_I \in \mathbb{Z}_+^I, \; x_T \in \mathbb{R}_+^T
\]

Then

- \( \nu(d) = z(d) \) for all \( d \in \mathbb{R} \).
- The variables in \( C \setminus T \) are redundant.
- \( z \) can be represented with at most 2 continuous variables.
Example

\[
\begin{align*}
\min & \quad x_1 - 3/4 x_2 + 3/4 x_3 \\
\text{s.t} & \quad 5/4 x_1 - x_2 + 1/2 x_3 = b, x_1, x_2 \in \mathbb{Z}^+, x_3 \in \mathbb{R}^+.
\end{align*}
\]

\[\eta^C = 3/2, \quad \zeta^C = -\infty.\]

For each discontinuous point \(d_i\), we have \(d_i - (5/4 y^i_1 - y^i_2) = 0\) and each linear segment has the slope of \(\eta^C = 3/2\).
Jeroslow Formula

Let \( M \in \mathbb{Z}_+ \) be such that for any \( t \in T \), \( \frac{M_{a_j}}{a_t} \in \mathbb{Z} \) for all \( j \in I \).

Then there is a Gomory function \( g \) such that

\[
z(d) = \min_{t \in T} \left\{ g\left(\lfloor d \rfloor_t \right) + \frac{c_t}{a_t} (d - \lfloor d \rfloor_t) \right\}, \quad \lfloor d \rfloor_t = \frac{a_t}{M} \left\lfloor \frac{Md}{a_t} \right\rfloor, \quad \forall d \in \mathbb{R}
\]

Such a Gomory function can be obtained from the value function of a related PILP.

For \( t \in T \), setting

\[
\omega_t(d) = g\left(\lfloor d \rfloor_t \right) + \frac{c_t}{a_t} (d - \lfloor d \rfloor_t) \quad \forall d \in \mathbb{R},
\]

we can write

\[
z(d) = \min_{t \in T} \omega_t(d) \quad \forall d \in \mathbb{R}
\]
### Piecewise Linearity and Continuity

- For \( t \in T \), \( \omega_t \) is piecewise linear with finitely many linear segments on any closed interval and each of those linear segments has a slope of \( \eta^C \) if \( t = t^+ \) or \( \zeta^C \) if \( t = t^- \).
- \( \omega_t^+ \) is continuous from the right, \( \omega_t^- \) is continuous from the left.
- \( \omega_t^+ \) and \( \omega_t^- \) are both lower-semicontinuous.

### Theorem

- \( z \) is piecewise-linear with finitely many linear segments on any closed interval and each of those linear segments has a slope of \( \eta^C \) or \( \zeta^C \).
- (Meyer 1975) \( z \) is lower-semicontinuous.
- \( \eta^C < \infty \) if and only if \( z \) is continuous from the right.
- \( \zeta^C > -\infty \) if and only if \( z \) is continuous from the left.
- Both \( \eta^C \) and \( \zeta^C \) are finite if and only if \( z \) is continuous everywhere.
Maximal Subadditive Extension

- Let \( f : [0, h] \rightarrow \mathbb{R}, h > 0 \) be subadditive and \( f(0) = 0 \).
- The maximal subadditive extension of \( f \) from \([0, h]\) to \( \mathbb{R}_+ \) is

\[
f_S(d) = \begin{cases} 
    f(d) & \text{if } d \in [0, h] \\
    \inf_{C \in \mathcal{C}(d)} \sum_{\rho \in C} f(\rho) & \text{if } d > h
\end{cases}
\]

- \( \mathcal{C}(d) \) is the set of all finite collections \( \{\rho_1, \ldots, \rho_R\} \) such that \( \rho_i \in [0, h], i = 1, \ldots, R \) and \( \sum_{i=1}^{R} \rho_i = d \).
- Each collection \( \{\rho_1, \ldots, \rho_R\} \) is called an \( h \)-partition of \( d \).
- We can also extend a subadditive function \( f : [h, 0] \rightarrow \mathbb{R}, h < 0 \) to \( \mathbb{R}_- \) similarly.

- (Bruckner 1960) \( f_S \) is subadditive and if \( g \) is any other subadditive extension of \( f \) from \([0, h]\) to \( \mathbb{R}_+ \), then \( g \leq f_S \) (maximality).
Suppose we use $z$ itself as the seed function.

Observe that we can change the “$\inf$” to “$\min$”:

**Lemma**

Let the function $f : [0, h] \rightarrow \mathbb{R}$ be defined by $f(d) = z(d) \ \forall d \in [0, h]$. Then,

$$f_S(d) = \begin{cases} 
  z(d) & \text{if } d \in [0, h] \\
  \min_{C \in C(d)} \sum_{\rho \in C} z(\rho) & \text{if } d > h
\end{cases}$$

- For any $h > 0$, $z(d) \leq f_S(d) \ \forall d \in \mathbb{R}_+$.
- Observe that for $d \in \mathbb{R}_+$, $f_S(d) \rightarrow z(d)$ while $h \rightarrow \infty$.
- Is there an $h < \infty$ such that $f_S(d) = z(d) \ \forall d \in \mathbb{R}_+$?
Yes! For large enough $h$, maximal extension produces the value function itself.

**Theorem**

Let $d_r = \max\{a_i \mid i \in N\}$ and $d_l = \min\{a_i \mid i \in N\}$ and let the functions $f_r$ and $f_l$ be the maximal subadditive extensions of $z$ from the intervals $[0, d_r]$ and $[d_l, 0]$ to $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively. Let

$$F(d) = \begin{cases} f_r(d) & d \in \mathbb{R}_+ \\ f_l(d) & d \in \mathbb{R}_- \end{cases}$$

then, $z = F$.

**Outline of the Proof.**

- $z \leq F$: By construction.
- $z \geq F$: Using MILP duality, $F$ is dual feasible.

In other words, the value function is completely encoded by the breakpoints in $[d_l, d_r]$ and 2 slopes.
General Procedure

- We will construct the value function in two steps
  - Construct the value function on \([d_l, d_r]\).
  - Extend the value function to the entire real line from \([d_l, d_r]\).
- For the rest of the talk
  - We assume \(\eta^c < \infty\) and \(\zeta^c < \infty\).
  - We construct the value function over \(\mathbb{R}_+\) only.
  - These assumptions are only needed to simplify the presentation.
Constructing the Value Function on $[0, d_r]$

- If both $\eta^C$ and $\zeta^C$ are finite, the value function is continuous and the slopes of the linear segments alternate between $\eta^C$ and $\zeta^C$.
- For $d_1, d_2 \in [0, d_r]$, if $z(d_1)$ and $z(d_2)$ are connected by a line with slope $\eta^C$ or $\zeta^C$, then $z$ is linear over $[d_1, d_2]$ with the respective slope (subadditivity).
- With these observations, we can formulate a finite algorithm to evaluate $z$ in $[d_l, d_r]$. 
Example (cont’d)

\[ d_r = 6: \]

**Figure:** Evaluating \( z \) in \([0, 6]\)
Example (cont’d)

\[ d_r = 6: \]

\[ \eta^C \quad \zeta^C \]

\[ \begin{array}{c}
0 \\
2 \\
4 \\
6 \\
8 \\
\end{array} \]

\[ \begin{array}{c}
0 \\
2 \\
4 \\
6 \\
\end{array} \]

\textbf{Figure: Evaluating } z \text{ in } [0, 6]
Example (cont’d)

\[ d_r = 6: \]

\[ \eta^C \quad \xi^C \]

**Figure:** Evaluating \( z \) in \([0, 6]\)
Example (cont’d)

\[ d_r = 6 : \]

\[ \eta^C - \zeta^C \]

Figure: Evaluating \( z \) in \([0, 6]\)
Example (cont’d)

\[ d_r = 6: \]

Figure: Evaluating \( z \) in \([0, 6]\)
Extending the Value Function

Consider evaluating

$$z(d) = \min_{C \in C(d)} \sum_{\rho \in C} z(\rho) \text{ for } d \not\in [0, d_r].$$

- Can we limit $|C|$, $C \in C(d)$? Yes!
- Can we limit $|C(d)|$? Yes!

**Theorem**

Let $d > d_r$ and let $k_d \geq 2$ be the integer such that $d \in \left(\frac{kd}{2}d_r, \frac{kd+1}{2}d_r\right]$. Then

$$z(d) = \min\left\{ \sum_{i=1}^{kd} z(\rho_i) \mid \sum_{i=1}^{kd} \rho_i = d, \rho_i \in [0, d_r], i = 1, \ldots, kd \right\}.$$

- Therefore, $|C| \leq k_d$ for any $C \in C(d)$.
- How about $|C(d)|$?
Let $\Psi$ be the lower break points of $z$ in $[0, d_r]$.

**Theorem**

For any $d \in \mathbb{R}_+ \setminus [0, d_r]$ there is an optimal $d_r$-partition $C \in C(d)$ such that $|C \setminus \Psi| \leq 1$.

- In particular, we only need to consider the collection

$$\Lambda(d) \equiv \{\mathcal{H} \cup \{\mu\} \mid \mathcal{H} \in C(d - \mu) \cap \Psi^{k_d-1}, \sum_{\rho \in \mathcal{H}} \rho + \mu = d, \mu \in [0, d_r]\}$$

In other words,

$$z(d) = \min_{C \in \Lambda(d)} \sum_{\rho \in C} z(\rho) \quad \forall d \in \mathbb{R}_+ \setminus [0, d_r]$$

- Observe that the set $\Lambda(d)$ is finite.
Example (cont’d)

For the interval $[0, 6]$, we have $\Psi = \{0, 5, 6\}$. For $b = \frac{31}{2}$, $C = \{5, 5, \frac{11}{2}\}$ is an optimal $d_r$-partition with $|C \setminus \Psi| = 1$. 

![Diagram showing the value function with points and line segments indicating the optimization process.](chart)
Getting $z$ over $\mathbb{R}_+$

- **Recursive Construction:**
  - Let $\Psi((0, p])$ to the set of the lower break points of $z$ in the interval $(0, p] \ p \in \mathbb{R}_+$.
  - Let $p := d_r$.
  - For any $d \in (p, p + \frac{p}{2}]$, let
    \[
    z(d) = \min\{z(\rho_1) + z(\rho_2) \mid \rho_1 + \rho_2 = d, \rho_1 \in \Psi((0, p]), \rho_2 \in (0, p]\}
    \]
  - Let $p := p + \frac{p}{2}$ and repeat this step.
In other words, we do the following at each iteration:

$$z(d) = \min_{j} g^j(d) \quad \forall d \in \left(p, p + \frac{p}{2}\right]$$

where, for each $d^j \in \Psi((0, p])$, the functions $g^j : [0, p + \frac{p}{2}] \to \mathbb{R} \cup \{\infty\}$ are defined as

$$g^j(d) = \begin{cases} 
    z(d) & \text{if } d \leq d^j, \\
    z(d^j) + z(d - d^j) & \text{if } d^j < d \leq p + d^j, \\
    \infty & \text{otherwise.}
\end{cases}$$

Because of subadditivity, we can then write

$$z(d) = \min_{j} g^j(d) \quad \forall d \in \left(0, p + \frac{p}{2}\right].$$
Example (cont’d)

Extending the value function of (SP) from $[0, 6]$ to $[0, 9]$
Extending the value function of (SP) from \([0, 6]\) to \([0, 9]\)
Example (cont’d)

Extending the value function of (SP) from $[0, 9]$ to $[0, \frac{27}{2}]$
Example (cont’d)

Extending the value function of (SP) from $[0, 9]$ to $[0, \frac{27}{2}]$
Observe that it is enough to get the **lower break points** and this can be done more easily.

**Theorem**

*If \( d \) is a lower break-point of \( z \) on \((p, p + \frac{p}{2}]\) then there exist \( \rho_1, \rho_2 \in \Psi((0, p]) \) such that \( z(d) = z(\rho_1) + z(\rho_2) \) and \( d = \rho_1 + \rho_2 \).*

- Set \( \Upsilon(p) \equiv \{ z(\rho_1) + z(\rho_2) \mid p < \rho_1 + \rho_2 \leq p + \frac{p}{2}, \rho_1, \rho_2 \in \Psi((0, p]) \} \).
- Then, \( z \) is obtained by connecting the points on the “**lower envelope**” of \( \Upsilon(p) \).
- Can we make the procedure finite?
Termination

Yes! Periodicity

- Let $\mathcal{D} = \{d \mid z(d) = F_L(d)\}$. Note that $\mathcal{D} \neq \emptyset$.
- Furthermore, let $\lambda = \min\{d \mid d \geq d_r, d \in \mathcal{D}\}$.
- Define the functions $f_j : \mathbb{R}_+ \to \mathbb{R}, j \in \mathbb{Z}_+ \setminus \{0\}$ as follows

$$f_j(d) = \begin{cases} z(d) & , d \leq j \lambda \\ kz(\lambda) + z(d - k \lambda) & , d \in ((k + j - 1) \lambda, (k + j) \lambda], k \in \mathbb{Z}_+ \setminus \{0\}. \end{cases}$$

Theorem

1. $f_j(d) \geq f_{j+1}(d) \geq z(d)$ for all $d \in \mathbb{R}_+, j \in \mathbb{Z}_+ \setminus \{0\}$.
2. There exists $q \in \mathbb{Z}_+ \setminus \{0\}$ such that $z(d) = f_q(d)$ $\forall d \in \mathbb{R}_+$.
3. In addition, $z(d) = f_q(d)$ $\forall d \in \mathbb{R}_+$ if and only if $f_q(d) = f_{q+1}(d)$ $\forall d \in \mathbb{R}_+$.

Therefore, we can extend over the intervals of size $\lambda$ and stop when we reach the 3. condition above.
Example (cont’d)

\[ \lambda = 6, \quad f_1(d) = \begin{cases} 
  z(d) & d \leq 6 \\
  k z(6) + z(d - 6k) & d \in (6k, 6(k + 1)], \quad k \in \mathbb{Z}_+ \setminus \{0\}.
\]
Example (cont’d)

\[ f_2(d) = \begin{cases} 
  z(d) & , d \leq 12 \\
  kz(6) + z(d - 6k) & , d \in (6(k + 1), 6(k + 2)], k \in \mathbb{Z}_+ \setminus \{0\}.
\end{cases} \]
Example (cont’d)

\[ f_3(d) = \begin{cases} 
  z(d) & , d \leq 18 \\
  k z(6) + z(d - 6k) & , d \in (6(k + 2), 6(k + 3)], k \in \mathbb{Z}_+ \setminus \{0\}. 
\end{cases} \]
Example (cont’d)

\[ f_4(d) = \begin{cases} 
    z(d), & d \leq 24 \\
    kz(6) + z(d - 6k), & d \in (6(k + 3), 6(k + 4)], k \in \mathbb{Z}_+ \setminus \{0\}.
\end{cases} \]

Note that \( f_4(d) = f_5(d) \forall d \in \mathbb{R}_+ \). Therefore, \( z(d) = f_4(d) \forall d \in \mathbb{R}_+ \).
A Finite Procedure

We can further restrict the search space by again using maximal extension and the fact that $z(k\lambda) = kz(\lambda)$ and $\lambda \geq d_r$.

**Theorem**

For a given $k \geq 2$, $k \in \mathbb{Z}_+$,

$$z(d) = \min\{z(\rho_1) + z(\rho_2) \mid \rho_1 + \rho_2 = d, \rho_1 \in (0, 2\lambda], \rho_2 \in ((k - 1)\lambda, k\lambda]\}$$

$\forall d \in (k\lambda, (k + 1)\lambda]$.

- **Revised Recursive Construction:**
  1. Let $p := 2\lambda$.
  2. Set $\Upsilon(p) \equiv \{z(\rho_1) + z(\rho_2) \mid p < \rho_1 + \rho_2 \leq p + \lambda, \rho_1 \in \Psi((0, 2\lambda]), \rho_2 \in \Psi((p - \lambda, p])\}$ and obtain $z$ over $[p, p + \lambda]$ by considering the “lower subadditive envelope” of $\Upsilon(p)$.
  3. If $z(d) = z(d - \lambda) + z(\lambda) \forall d \in \Psi((p, p + \lambda))$, then stop. Otherwise, let $p := p + \lambda$ and repeat the last step.
Extending the value function of (SP) from \([0, 12]\) to \([0, 18]\)
Example (cont’d)

Extending the value function of (SP) from $[0, 12]$ to $[0, 18]$
Example (cont’d)

Extending the value function of (SP) from $[0, 18]$ to $[0, 24]$. 

![Graph showing the value function extension from $[0, 18]$ to $[0, 24]$]
Example (cont’d)

Extending the value function of (SP) from $[0, 18]$ to $[0, 24]$
Consider a general mixed integer linear program (MILP)

\[ z_P = \min_{x \in S} cx, \quad (P) \]

c, \in \mathbb{R}^n, \quad S = \{x \in \mathbb{Z}_+^r \times \mathbb{R}^{n-r}_+ \mid Ax = b\} \text{ with } A \in \mathbb{Q}^{m \times n}, b \in \mathbb{R}^m.

The value function of the primal problem (P) is

\[ z(d) = \min_{x \in S(d)} cx, \]

where for a given \( d \in \mathbb{R}^m \), \( S(d) = \{x \in \mathbb{Z}_+^r \times \mathbb{R}^{n-r}_+ \mid Ax = d\} \).
Let the set $\mathcal{E}$ consist of the index sets of dual feasible bases of the linear program

$$\min\left\{ \frac{1}{M} c_{C} x_{C} : \frac{1}{M} A_{C} x_{C} = b, x \geq 0 \right\}$$

where $M \in \mathbb{Z}_{+}$ such that for any $E \in \mathcal{E}$, $MA_{E}^{-1}a^{j} \in \mathbb{Z}^{m}$ for all $j \in I$.

**Theorem (Jeroslow Formula)**

There is a $g \in \mathcal{G}^{m}$ such that

$$z(d) = \min_{E \in \mathcal{E}} g(\lfloor d \rfloor_{E}) + v_{E}(d - \lfloor d \rfloor_{E}) \quad \forall d \in \mathbb{R}^{m} \text{ with } S(d) \neq \emptyset,$$

where for $E \in \mathcal{E}$, $\lfloor d \rfloor_{E} = A_{E} \lfloor A_{E}^{-1} d \rfloor$ and $v_{E}$ is the corresponding basic feasible solution.
For $E \in \mathcal{E}$, setting

$$
\omega_E(d) = g(\lfloor d \rfloor_E) + v_E(d - \lfloor d \rfloor_E) \quad \forall d \in \mathbb{R}^m \text{ with } S(d) \neq \emptyset,
$$

we can write

$$
z(d) = \min_{E \in \mathcal{E}} \omega_E(d) \quad \forall d \in \mathbb{R}^m \text{ with } S(d) \neq \emptyset.
$$

Many of our previous results can be extended to general case in the obvious way.

Similarly, we can use maximal subadditive extensions to construct the value function.

However, an obvious combinatorial explosion occurs.

Therefore, we consider using single row relaxations to get a subadditive approximation.
Basic Idea

Consider the value functions of each single row relaxation:

\[ z_i(q) = \min \{ cx \mid a_i x = q, x \in \mathbb{Z}_+^r \times \mathbb{R}_+^{n-r} \} \quad q \in \mathbb{R}, i \in M \equiv \{1, \ldots, m\} \]

where \( a_i \) is the \( i^{th} \) row of \( A \).

Theorem

Let \( F(d) = \max_{i \in M} \{ z_i(d_i) \} \), \( d = (d_1, \ldots, d_m) \), \( d \in \mathbb{R}^m \). Then \( F \) is subadditive and \( z(d) \geq F(d) \ \forall d \in \mathbb{R}^m \).
Maximal Subadditive Extension

Assume that $A \in \mathbb{Q}^m_+$. Let $S \subseteq M$ and $q^r \in \mathbb{Q}^{|S|}_+$ be the vector of the maximum of the coefficients of rows $a_i, i \in S$. Define

$$G_S(q) = \begin{cases} 
\max_{i \in S} \{z_i(q_i)\} & q_i \in [0, q^r_i] \ i \in M \\
\max \left\{ \max_{i \in K} \{z_i(q_i)\}, \inf_{C \in \mathcal{C}(q_S \setminus K)} \sum_{\rho \in C} G_S(\rho) \right\} & q_i \in [0, q^r_i] \ i \in K \\
\inf_{C \in \mathcal{C}(q)} \sum_{\rho \in C} G_S(\rho) & q_i > q^r_i \ i \in S \setminus K \\
& K \subseteq S \\
& q_i \in \mathbb{R}_+ \setminus [0, q^r_i] \ i \in M
\end{cases}$$

for all $q \in \mathbb{R}^{|S|}$ where for $T \subseteq S$, $\mathcal{C}(q_T)$ is the set of all finite collections $\{\rho_1, \ldots, \rho_R\}, \rho_j \in \mathbb{R}^{|T|}$ such that $\rho_j \in \times_{i \in T}[0, q^r_i], j = 1, \ldots, R$ and $\sum_{j=1}^R \rho_j = q_T$. 

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Maximal Subadditive Extension

$G_S$ is simply the maximal subadditive extension of the function
\[
\max_{i \in S} \{ z_i(q_i) \}
\]
from the box \( \times_{i \in S} [0, q_i^r] \) to \( \mathbb{R}^{|S|}_+ \).

**Theorem**

Let \( F_S(d) = \max \left\{ G_S(d_S), \max_{i \in M \setminus S} \{ z_i(d_i) \} \right\} \). \( F_S \geq \max_{i \in M} \{ z_i(d_i) \} \), is subadditive and \( z(d) \geq F_S(d) \) for all \( d \in \mathbb{R}^m_+ \).
Aggregation

For $S \subseteq M$, $\omega \in \mathbb{R}^{|S|}$, set

$$G_S(q, \omega) = \min \{ cx \mid \omega a_s x = \omega q, x \in \mathbb{Z}_+^r \times \mathbb{R}_+^{n-r} \} \forall q \in \mathbb{R}^{|S|}$$

**Theorem**

Let

$$F_S(\omega, d) = \max \left\{ G_S(d_S, \omega), \max_{i \in M \setminus S} \{ z_i(d_i) \} \right\}, \quad d \in \mathbb{R}^m.$$

$F_S$ is subadditive and $z(d) \geq F_S(\omega, d)$ for any $\omega \in \mathbb{R}^{|S|}$, $d \in \mathbb{R}^m$.

As with cutting planes, different aggregation procedures are possible.
Using Cuts

- Assume that $S(d) = \{ x \in \mathbb{Z}_+^n : Ax \leq d \}$.
- Consider the set of Gomory cuts $\Pi x \geq \Pi^0$, $\Pi \in \mathbb{Q}^{k \times n}$, $\Pi^0 \in \mathbb{Q}^k$ defined by the sets of multipliers $\Omega = \{ \omega^1, \ldots, \omega^{k-1} \}$, $\omega^i \in \mathbb{Q}_{+}^{m+i-1}$ as follows

$$
\Pi_{ij} = \left[ \sum_{l=1}^{m} \omega^i_l A_{lj} + \sum_{l=1}^{i-1} \omega^i_{m+l} \Pi^0_l \right] \quad \forall i = 1, \ldots, k, j = 1, \ldots, n
$$

$$
\Pi_i^0 = \left[ \sum_{l=1}^{m} \omega^i_l d_l + \sum_{l=1}^{i-1} \omega^i_{m+l} \Pi^0_l \right] \quad \forall i = 1, \ldots, k
$$

**Theorem**

For $\Omega = \{ \omega^1, \ldots, \omega^{k-1} \}$, $\omega^i \in \mathbb{Q}_{+}^{m+i-1}$, $k \in \mathbb{Z}_+$, let $z_{m+i}(\omega^i, d)$ denote the value function of row $m+i$, $i = 1, \ldots, k-1$ and

$$
F(\Omega, d) = \max \left\{ \max_{i \in M} z(d_i), \max_{i=1, \ldots, k-1, \omega^i \in \Omega} z_{m+i}(\omega^i, d) \right\}.
$$

Then, $F$ is subadditive and $z(d) \geq F(\Omega, d)$ for any $d \in \mathbb{R}^m$. 
Current and Future Work

- Extending the theory and algorithms to the general case.
- Developing upper bounding approximations.
- Integrating these procedures in with applications
  - Bilevel programming
  - Combinatorial auctions
- Answering the question

“Can we do anything practical with any of this?”