Biobjective Integer Programming

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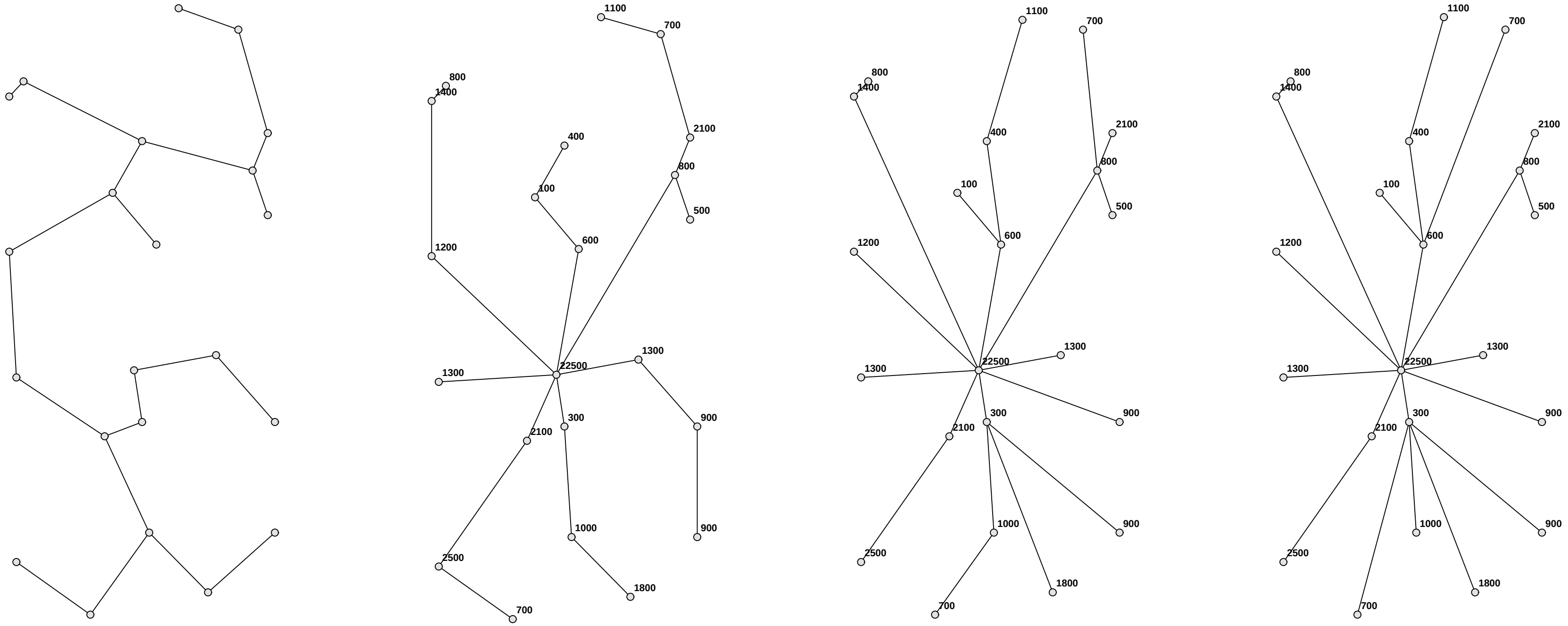


The SAS Institute, Thursday, May 18, 2006

Outline of Talk

- Preliminaries
- The WCN Algorithm
- Variants
  - Interactive algorithm
  - Approximation algorithm
- Enhancements
  - Avoiding weakly dominated solutions
  - Improving efficiency
- Examples and Applications
  - Parametric Programming
  - Network Routing
- Computational Results

Biobjective Mixed-integer Programs

A *biobjective* or *bicriterion mixed-integer program* (BMIP) is an optimization problem of the form

$$\begin{array}{ll} \text{vmax} & f(x) \\ \text{subject to} & x \in X, \end{array}$$

where

- $f : \mathbb{R}^n \to \mathbb{R}^2$ is the *(bicriteria) objective function*, and
- $X \subset \mathbb{Z}^p \times \mathbb{R}^{n-p}$ is the *feasible region*, usually defined to be

$$\{x \in \mathbb{Z}^p \times \mathbb{R}^{n-p} \mid g_i(x) \leq 0, i = 1, \ldots, m\}$$

for functions $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$.

The vmax operator indicates that the goal is to generate the set of *efficient solutions* (defined next).

Some Definitions

- We define the set of *outcomes* to be $Y = f(X) \subset \mathbb{R}^2$.
- In outcome space, BMIP can be restated as

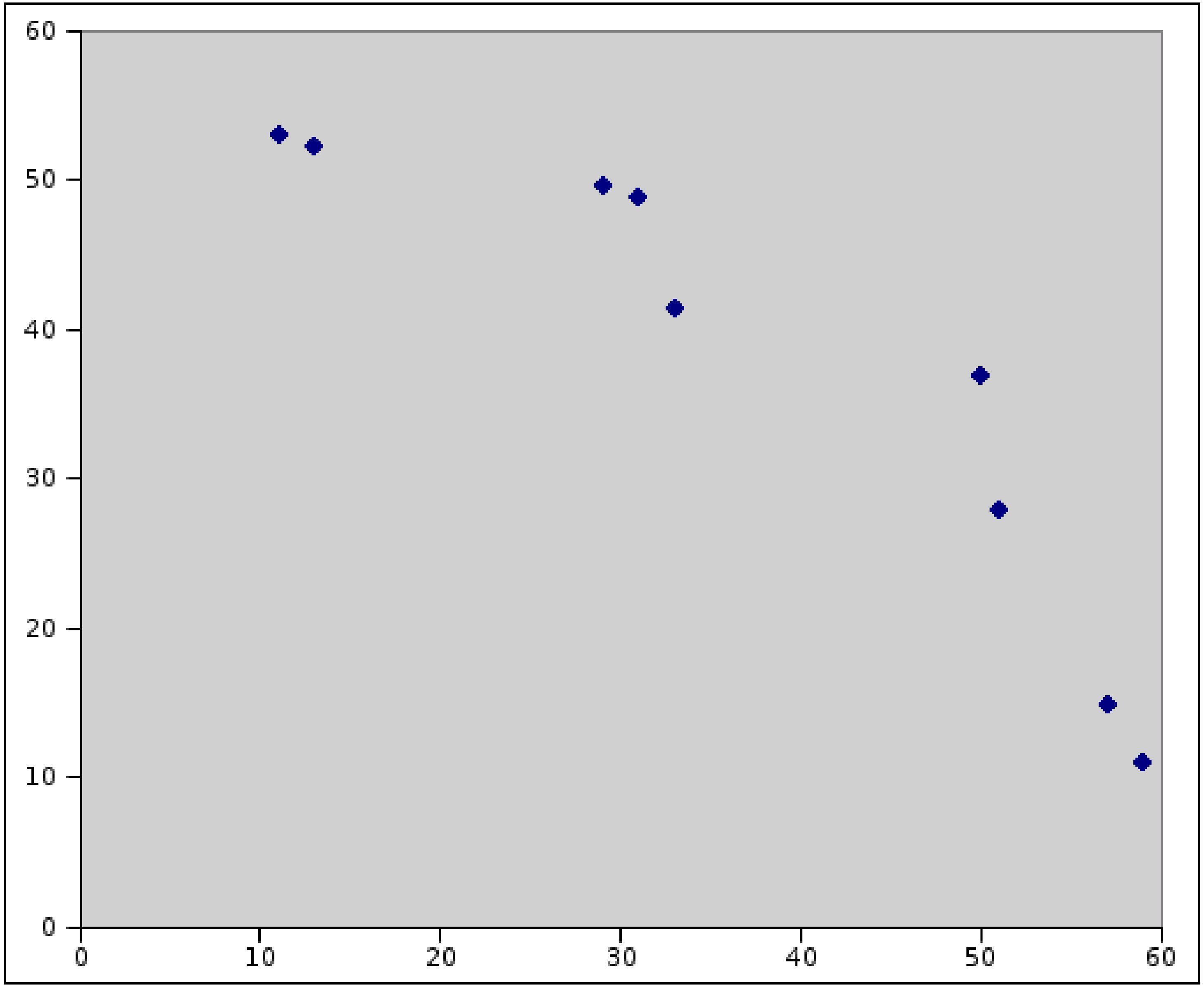
$$\begin{array}{ll} \text{vmax} & y \\ \text{subject to} & y \in f(X), \end{array}$$

- For convenience, we will work primarily in outcome space.
- $x^1 \in X$ *dominates* $x^2 \in X$ if $f_i(x_1) \geq f_i(x_2)$ for $i = 1, 2$ and at least one inequality is strict.
- If both inequalities are strict the dominance is *strong* (otherwise *weak*).
- Any $x \in X$ not dominated by another member of X is said to be *efficient*.
- If $x \in X$ is efficient, then $y = f(x)$ is a *Pareto outcome*.
- Our goal is to generate the set of all Pareto outcomes.

More Definitions

- We will denote the set of efficient solutions by X_E.
- The set of Pareto outcomes is then $Y_E = f(X_E)$.
- We assume that $|Y_E|$ is finite.
- If $x \in X_E$ strongly dominates all members of $X \setminus X_E$, then x is said to be *strongly efficient*.
- Likewise, if $x \in X_E$ is strongly efficient, then $y = f(x)$ is *strongly Pareto*.
- If all members of Y_E are strongly Pareto, then Y_E is said to be *uniformly dominant*.
- The assumption of uniform dominance simplifies computation substantially, but is not satisfied in most practical settings.

Illustrating Pareto Outcomes



Algorithms for Generating Pareto Outcomes

- A number of algorithms for generating Pareto outcomes have been proposed.
- These can be categorized in several ways:
  - By output: complete enumeration, partial enumeration, or heuristic enumeration of Y_E.
  - By user interaction: Interactive or non-interactive.
  - By methodology: branch and bound, dynamic programming, implicit enumeration, weighted sums, weighted norms, probing.
- We present an algorithm
  - that can either partially or completely enumerate the Pareto set,
  - has both interactive and non-interactive variants,
  - is based on a modified branch and bound algorithm.

Probing Algorithms

- We will focus on *probing algorithms* that *scalarize* the objective, i.e., replace it with a single criterion.
- Such algorithms reduce solution of a BMIP to a series of MIPs.
- The main factor in the running time is then the number of probes.
- The most obvious scalarization is the *weighted sum objective*.
- We replace the original objective with

$$\max_{y \in f(X)} \beta y_1 + (1 - \beta) y_2$$

to obtain a parameterized family of MIPs.

Supported Outcomes

- Optimal solutions to weighted sum MIPs are extreme points of $conv(Y_E)$.
- Such outcomes are called *supported outcomes*.
- The set of all supported outcomes can easily be generated by solving a sequence of MIPs.
- Every supported outcome is Pareto, but the converse is not true.
- This makes it difficult as a tool to generate all Pareto outcomes.
- Chalmet (1986) suggested restricting the subproblems so that each Pareto outcome is supported on some subregion.
- Using this technique, it is possibe to generate all Pareto outcomes.

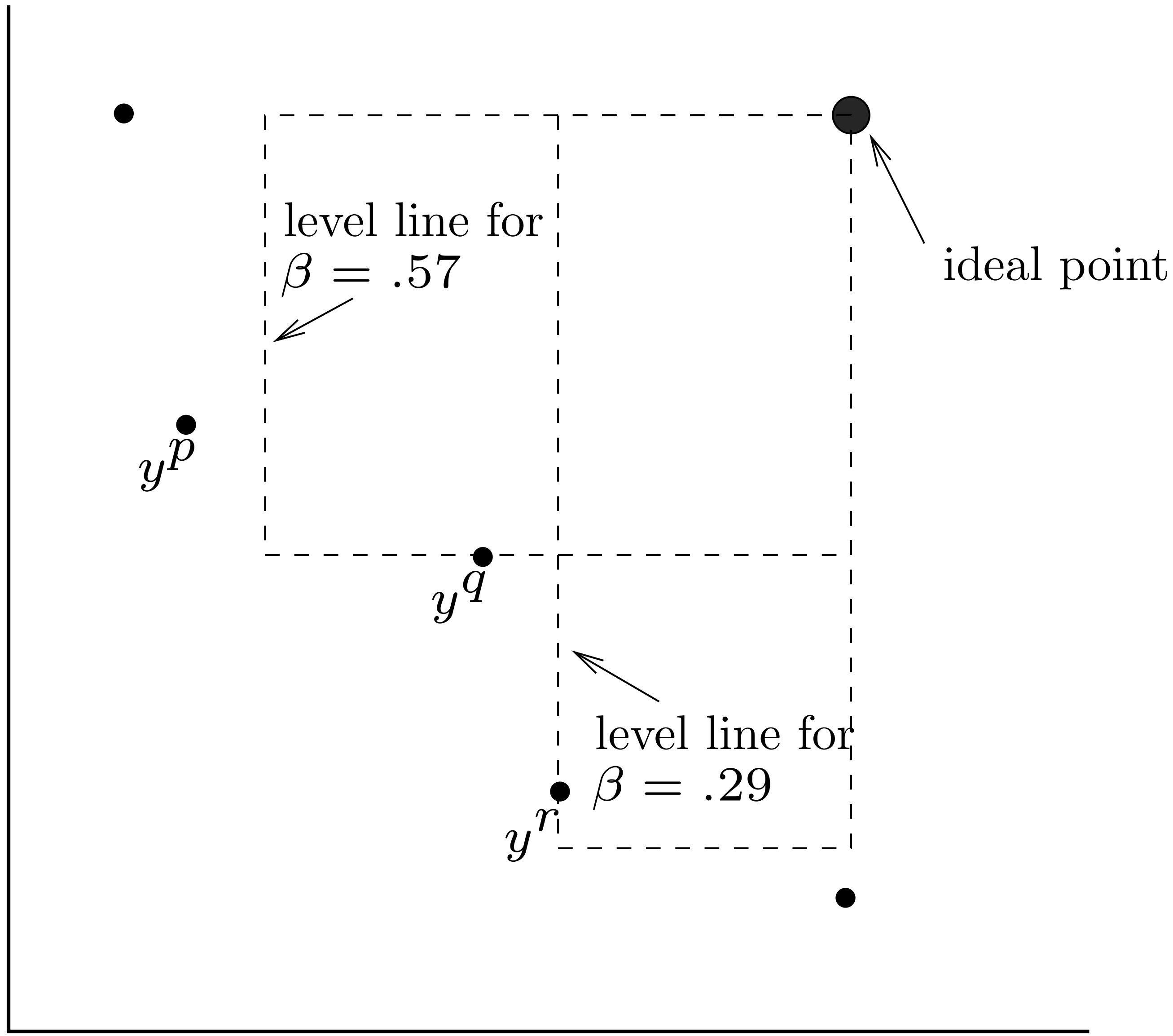
The Weighted Chebyshev Norm

- Another option is to replace the weighted sum objective with a *weighted Chebyshev norm* (WCN) objective.
- The *Chebyshev norm* (l_∞ norm) in $\mathbb{R}^2$ is defined by $\|y\|_\infty = \max\{|y_1|, |y_2|\}$.
- The *weighted Chebyshev norm* with weight $0 \leq \beta \leq 1$ is defined by $\|y\|_\infty = \max\{\beta|y_1|, (1-\beta)|y_2|\}$.
- The *ideal point* y^* is (y_1^*, y_2^*) where $y_i^* = \max_{x \in X}(f(x))_i$.
- Methods based on the WCN select outcomes with minimum WCN distance from the ideal point by solving

$$\min_{y \in f(X)} \{\|y^* - y\|_\infty^\beta\}. \tag{1}$$

- Bowman (1976) showed that every Pareto outcome is a solution to (1) for some $0 \leq \beta \leq 1$.
- The converse only holds if Y_E is uniformly dominant.

Illustrating the WCN



Ordering the Pareto Outcomes

- Eswaran (1989) suggested ordering the Pareto outcomes so that
  - $Y_E = \{y_p \mid 1 \leq p \leq N\}$, and
  - if $p < q$, then $y_1^p < y_1^q$ (and hence $y_2^p > y_2^q$).
- For any Pareto outcome y_p, if we define

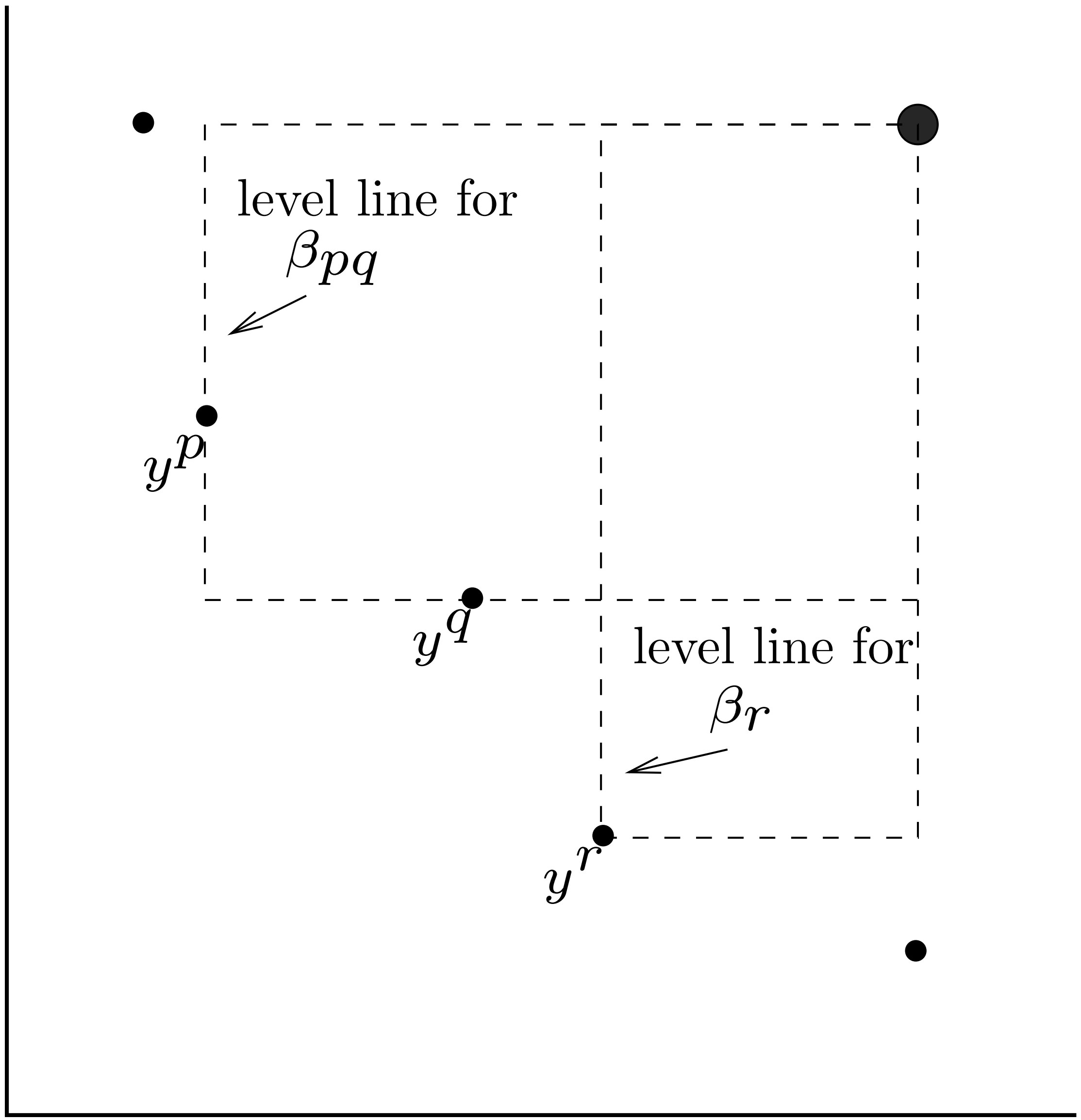
$$\beta_p = (y_2^* - y_2^p)/(y_1^* - y_1^p + y_2^* - y_2^p),$$

  then y^p is the unique optimal outcome for (1) with $\beta = \beta_p$.
- For any pair of Pareto outcomes y^p and y^q with $p < q$, if we define

$$\beta_{pq} = (y_2^* - y_2^q)/(y_1^* - y_1^p + y_2^* - y_2^q), \tag{2}$$

  then y^p and y^q are both optimal outcomes for (1) with $\beta = \beta_{pq}$.
- This provides us with a notion of *adjacency* and *breakpoints*.

Breakpoints Between Pareto Outcomes with the WCN



Algorithms Based on the WCN

- Solanki (1991) proposed an algorithm to generate an approximation to the Pareto set using the WCN.
  - The algorithm probes between pairs of known outcomes for new outcomes by restricting the domain ala Chalmet.
  - The search is controlled by an "error measure," which can be set to zero to get complete enumeration.
  - The number of probes is asymptotically optimal, but the algorithm does not produce breakpoints (directly).
- Eswaran (1989) proposed an algorithm based on binary search over the values of β.
  - In the worst case, the number of probes is

$$|Y_E|(1 - \lg(\xi(|Y_E| - 1))),$$

    where ξ is a chosen error parameter.
  - The algorithm produces only approximate breakpoint information.

The WCN Algorithm

Let $P(\beta)$ be the parameterized subproblem defined by (1) for a given weight β. The WCN algorithm is then:

Initialization Solve $P(1)$ and $P(0)$ to identify optimal outcomes y^1 and y^N, respectively, and the ideal point $y^* = (y_1^1, y_2^N)$. Set $I = \{(y^1, y^N)\}$.

Iteration While $I \neq \emptyset$ do:

1. Remove any (y^p, y^q) from I.
2. Compute β_{pq} as in (2) and solve $P(\beta_{pq})$. If the outcome is y^p or y^q, then y^p and y^q are adjacent in the list $(y^1, y^2, \ldots, y^N)$.
3. Otherwise, a new outcome y^r is generated. Add (y^p, y^r) and (y^r, y^q) to I.

This reduces solution of the original BMIP to solution of a sequence of $2N - 1$ MIPs, but still requires the assumption of uniform dominance.

Solving $P(\beta)$

- Problem (1) is equivalent to

$$\begin{array}{llll} \text{minimize} & z & & \\ \text{subject to} & z & \geq & \beta(y_1^* - y_1), \\ & z & \geq & (1-\beta)(y_2^* - y_2), \text{ and} \\ & y & \in & f(X). \end{array} \tag{3}$$

- This is a MIP, which can be solved by standard methods.
- This reformulation can still produce weakly dominated outcomes.

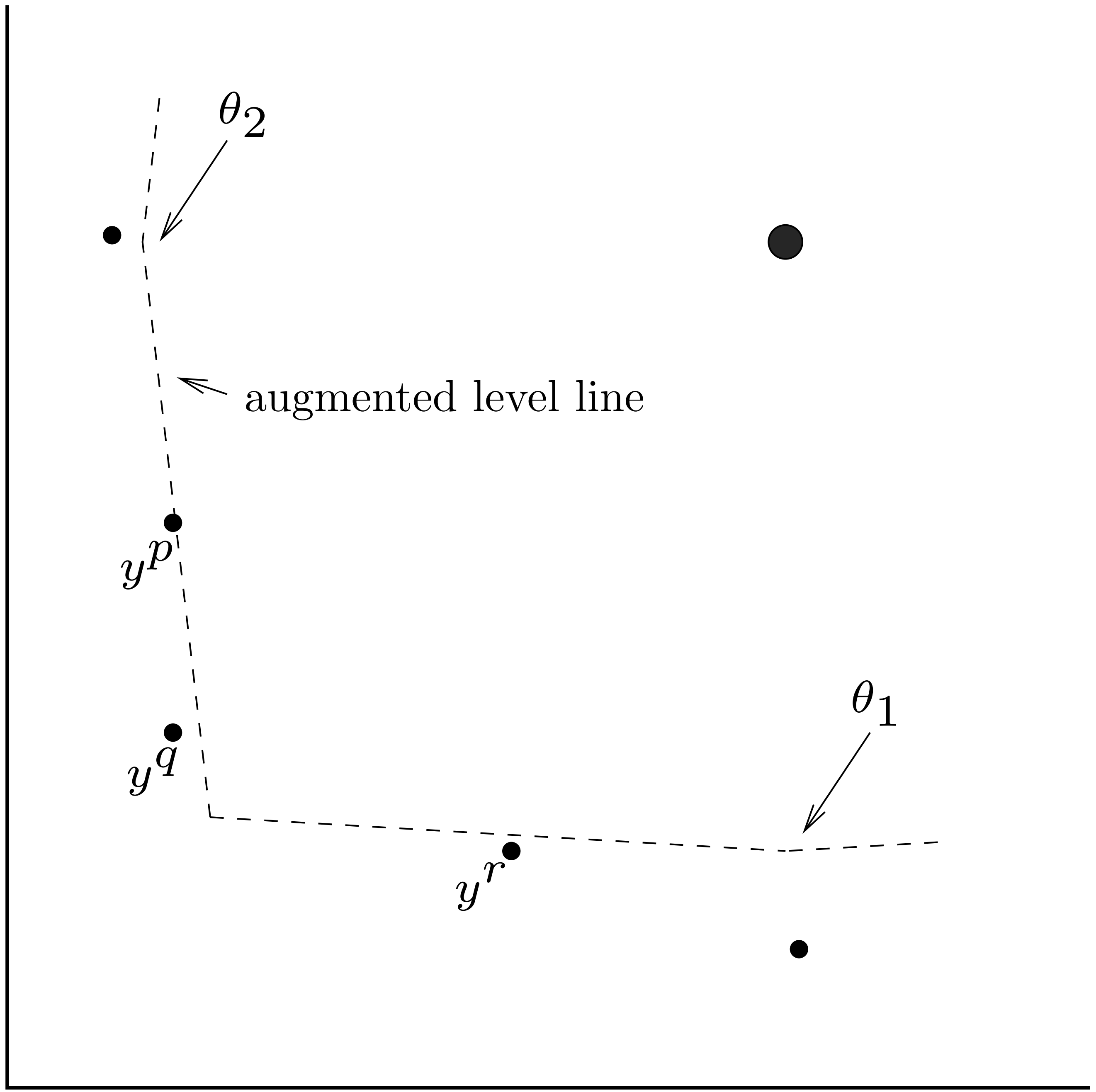
Relaxing the Uniform Dominance Requirement

- Dealing with weakly dominated outcomes is the most challenging aspect of these methods.
- We need a method of preventing $P(\beta)$ from producing weakly dominated outcomes.
- Weakly dominated outcomes are the same WCN distance from the ideal point as the outcomes they are dominated by.
- However, they are farther from the ideal point as measured by the l_p norm for $p < \infty$.
- One solution is to replace the WCN with the *augmented Chebyshev norm* (ACN), defined by

$$\|(y_1, y_2)\|_{\infty}^{\beta,\rho} = \max\{\beta|y_1|, (1-\beta)|y_2|\} + \rho(|y_1| + |y_2|),$$

where ρ is a small positive number.

Illustrating the ACN



Solving $P(\beta)$ with the ACN

- The problem of determining the outcome closest to the ideal point under this metric is

$$\begin{array}{llll} \min & z & + & \rho(|y_1^* - y_1| + |y_2^* - y_2|) \\ \text{subject to} & z & \geq & \beta(y_1^* - y_1) \\ & z & \geq & (1-\beta)(y_2^* - y_2) \\ & y & \in & f(X). \end{array} \tag{4}$$

- Because $y_k^* - y_k \geq 0$ for all $y \in f(X)$, the objective function can be rewritten as

$$\min z - \rho(y_1 + y_2).$$

- For fixed $\rho > 0$ small enough:
  - all optimal outcomes for problem (4) are Pareto (in particular, they are not weakly dominated), and
  - for a given Pareto outcome y for problem (4), there exists $0 \leq \hat{\beta} \leq 1$ such that y is the unique outcome to problem (4) with $\beta = \hat{\beta}$.
- In practice, choosing a proper value for ρ can be problematic.

Combinatorial Methods for Eliminating Weakly Dominated Solutions

- In the case of *biobjective linear integer programs* (BLIPs), we can employ combinatorial methods.
- Such a strategy involves implicitly enumerating alternative optimal solutions to $P(\beta)$.
- Weakly dominated outcomes are eliminated with cutting planes during the branch and bound procedure.
- Instead of pruning subproblems that yield feasible outcomes, we continue to search for alternative optima that dominate the current incumbant.
- To do so, we determine which of the two constraints

$$\begin{aligned} z &\geq \beta(y_1^* - y_1) \\ z &\geq (1-\beta)(y_2^* - y_2) \end{aligned}$$

from problem (1) is binding at $\hat{y}$.

Combinatorial Methods for Eliminating Weakly Dominated Solutions (cont'd)

- Let ϵ_1 and ϵ_2 be such that if y_r is a new outcome between y^p and y^q, then $y_i^r \geq \min\{y_i^p, y_i^q\} + \epsilon_i$, for $i = 1, 2$.
- If only the first constraint is binding, then the cut

$$y_1 \geq \hat{y}_1 + \epsilon_1$$

  is valid for any outcome that dominates $\hat{y}$.
- If only the second constraint is binding, then the cut

$$y_2 \geq \hat{y}_2 + \epsilon_2$$

  is valid for any outcome that dominates $\hat{y}$.
- If both constraints are binding, either cut can be imposed.

Hybrid Methods

- In practice, the ACN method is fast, but choosing the proper value of ρ is problematic.
- Combinatorial methods are less susceptible to numerical difficulties, but are slower.
- Combining the two methods improves running times and reduces dependence on the magnitude of ρ.

Other Enhancements to the Algorithm

- In Step 2, any new outcome y^r will have $y_1^r > y_1^p$ and $y_2^r > y_2^q$.
- If no such outcome exists, then the subproblem solver must still re-prove the optimality of y^p or y^q.
- Then it must be the case that

$$\|y^* - y^r\|_\infty^{\beta_{pq}} + \min\{\beta_{pq}\epsilon_1, (1-\beta_{pq})\epsilon_2\} \le \|y^* - y^p\|_\infty^{\beta_{pq}} = \|y^* - y^q\|_\infty^{\beta_{pq}}$$

- Hence, we can impose an a priori upper bound of

$$\|y^* - y^p\|_\infty^{\beta_{pq}} - \min\{\beta_{pq}\epsilon_1, (1-\beta_{pq})\epsilon_2\}$$

  when solving the subproblem $P(\beta_{pq})$.
- With this upper bound, each subproblem will either be infeasible or produce a new outcome.

Using Warm Starting

- We have been developing methodology for *warm starting* branch and bound computations.
- Because the WCN algorithm involves solving a sequence of slightly modified MILPs, warm starting can be used.
- Three approaches
  - Warm start from the result of the previous iteration.
  - Solve a "base" problem first and warm each subsequent problem from there.
  - Warm start from the "closest" previously solved subproblem.
- In addition, we can optionally save the global cut pool from iteration to iteration.

Approximating the Pareto Set

- If the number of Pareto outcomes is large, it may not be desirable to generate the entire set.
- If only part of the set is generated, it is important that the subset be *well-distributed* among the entire set.
- Any probing algorithm can generate an approximation to the Pareto set by terminating early.
  - In such case, the key is to avoid failed probes whenever possible.
  - The order in which the intervals are explored affects both the distribution of solutions and the number of failed probes.
  - Empirically, FIFO selection schemes tend to distribute the points well and also minimize the number of failed probes.
- Another approach is to generate the set of supported solutions.
- This can be an extremely bad approximation in some cases.

Interactive Algorithms

- Interactive algorithms offer another method of avoiding enumeration of the entire set.
- In an interactive algorithm, the user guides the solution process by providing real-time feedback.
- This feedback provides information about the user's unknown utility function.
- A simple feedback mechanism for the WCN algorithm is to allow the user to select the next interval to be explored.
- In this way, the user is able to zero in on the portion of the tradeoff curve that is most attractive.
- There are a number of mechanisms for providing estimated tradeoff information to the user as the algorithm progresses.

Implementation: A Brief Overview of SYMPHONY

- SYMPHONY is an open-source software package for solving and analyzing mixed-integer linear programs (MILPs).
- SYMPHONY can be used in three distinct modes.
  - Black box solver: Solve generic MILPs (command line or shell).
  - Callable library: Call SYMPHONY from a C/C++ code.
  - Framework: Develop a customized black box solver or callable library.
- Makes extensive use of the Computational Infrastructure for Operations Research (COIN-OR) libraries (`www.coin-or.org`).
- Complete documentation, code samples, data sets, and application plug-ins are available (`www.BranchAndCut.org`).
- Advanced features
  - Warm starting
  - Bicriteria solve
  - Sensitivity analysis
  - Parallel execution mode

Example: Bicriteria ILP

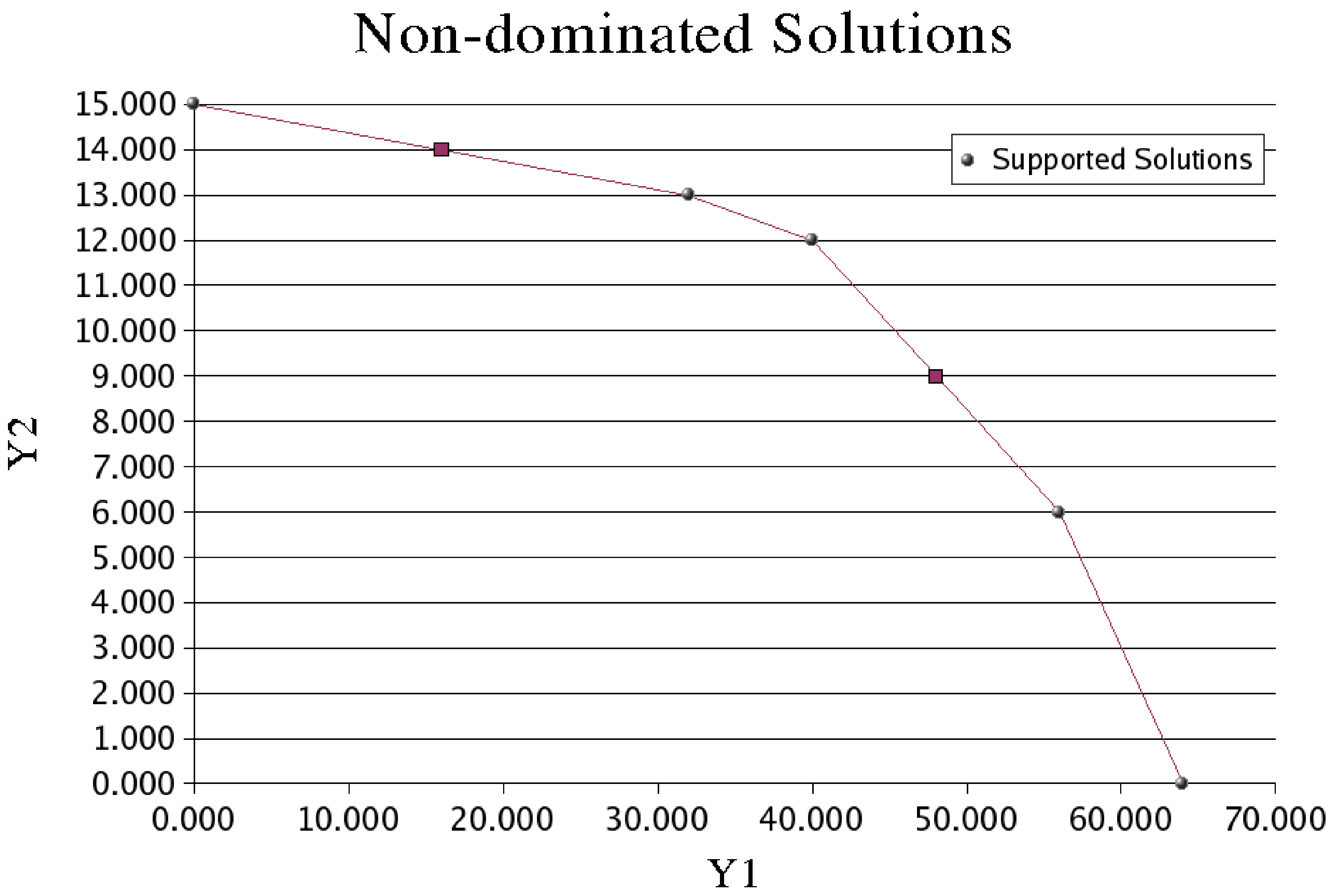
- Consider the following bicriteria ILP:

$$
\begin{aligned}
\text{vmax} \quad & [8x_1, x_2] \\
\text{s.t.} \quad & 7x_1 + x_2 \leq 56 \\
& 28x_1 + 9x_2 \leq 252 \\
& 3x_1 + 7x_2 \leq 105 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

- The following code solves this model using SYMPHONY.

```
int main(int argc, char **argv)
{
   OsiSymSolverInterface si;
   si.parseCommandLine(argc, argv);
   si.setObj2Coeff(1, 1);
   si.loadProblem();
   si.multiCriteriaBranchAndBound();
}
```

Example: Pareto Outcomes for Example

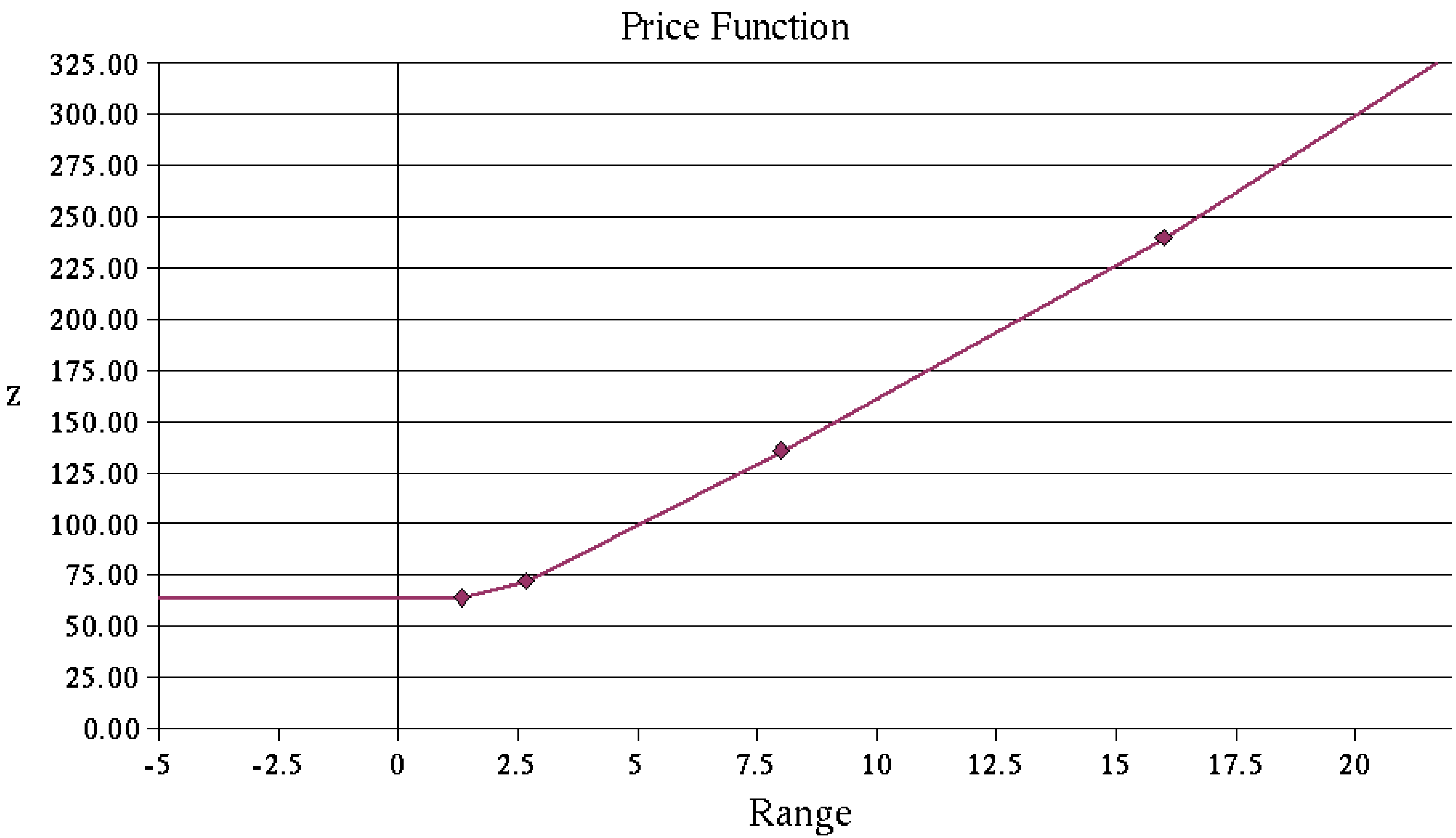


Example: Sensitivity Analysis

- By examining the supported solutions and break points, we can easily determine $p(\theta)$, the optimal solution to the ILP with objective $8x_1 + \theta x_2$.

θ range	$p(\theta)$	x_1^*	x_2^*
$(-\infty, 1.333)$	64	8	0
$(1.333, 2.667)$	$56 + 6\theta$	7	6
$(2.667, 8.000)$	$40+12\theta$	5	12
$(8.000, 16.000)$	$32+13\theta$	4	13
$(16.000, \infty)$	15θ	0	15

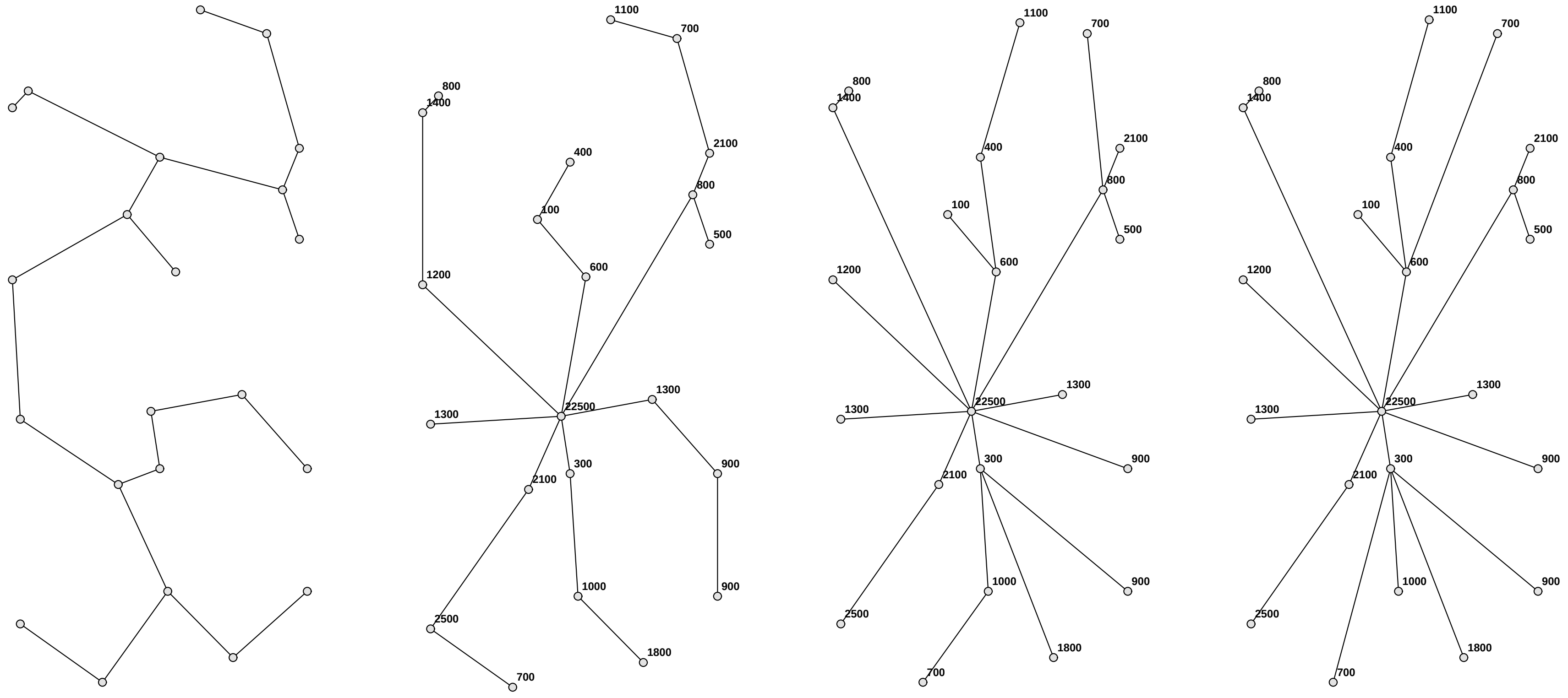
Example: Price Function



Application: Capacitated Network Routing Problems

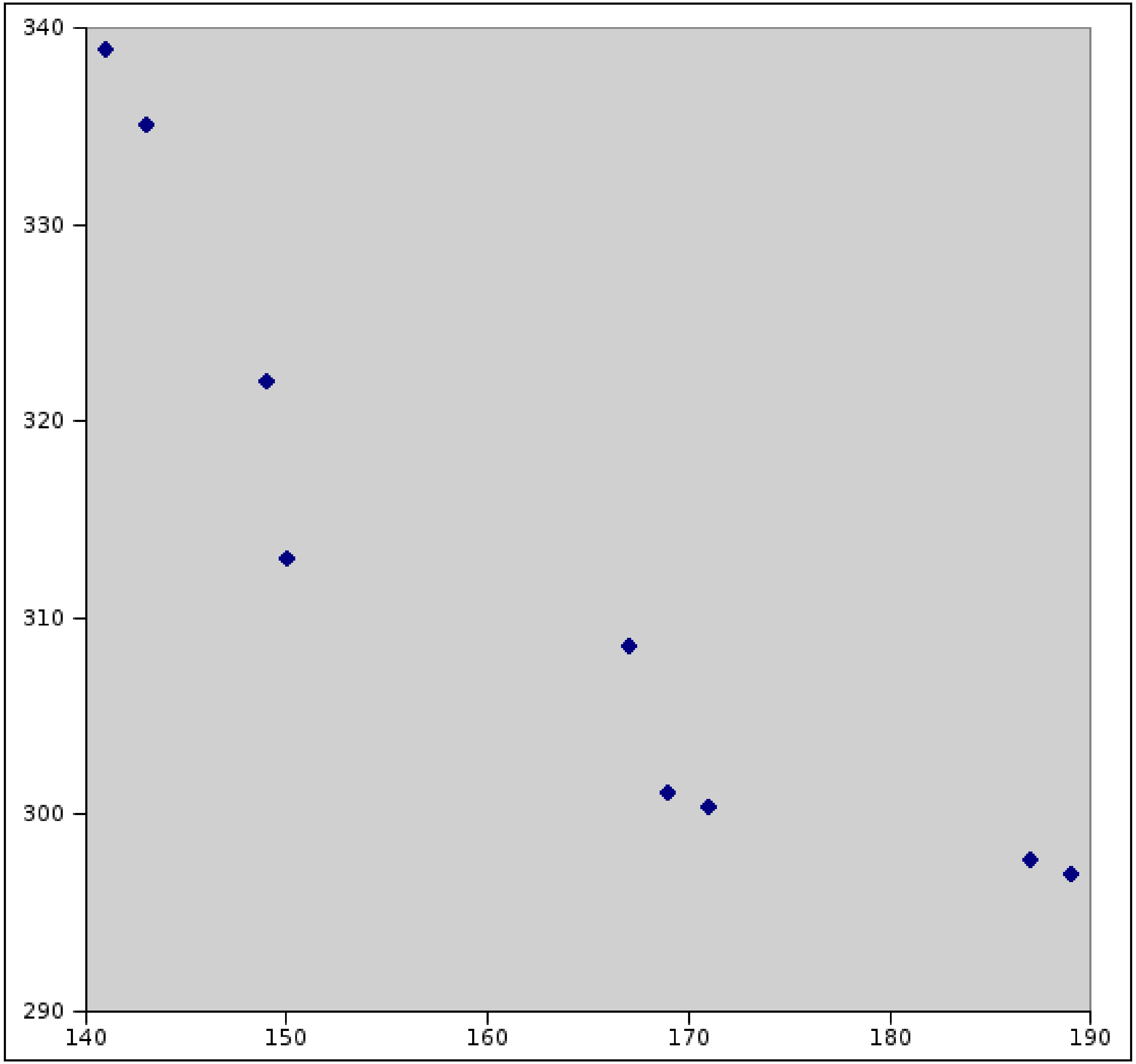
- Using SYMPHONY, we developed a custom solver for a class of capacitated network routing problems (CNRPs).
- A single commodity is supplied to a set of customers from a single supply point.
- We must design the network and route the demand, obeying capacity and other side constraints.
- We wish to consider both
  - the cost of construction (the sum of lengths of all links), and
  - the latency of the resulting network (the sum of length multiplied by demand carried for all links).
- These are competing objectives, so we can analyze the tradeoff by using the SYMPHONY multicriteria solver.

Application: Efficient Solutions for a Small CNRP

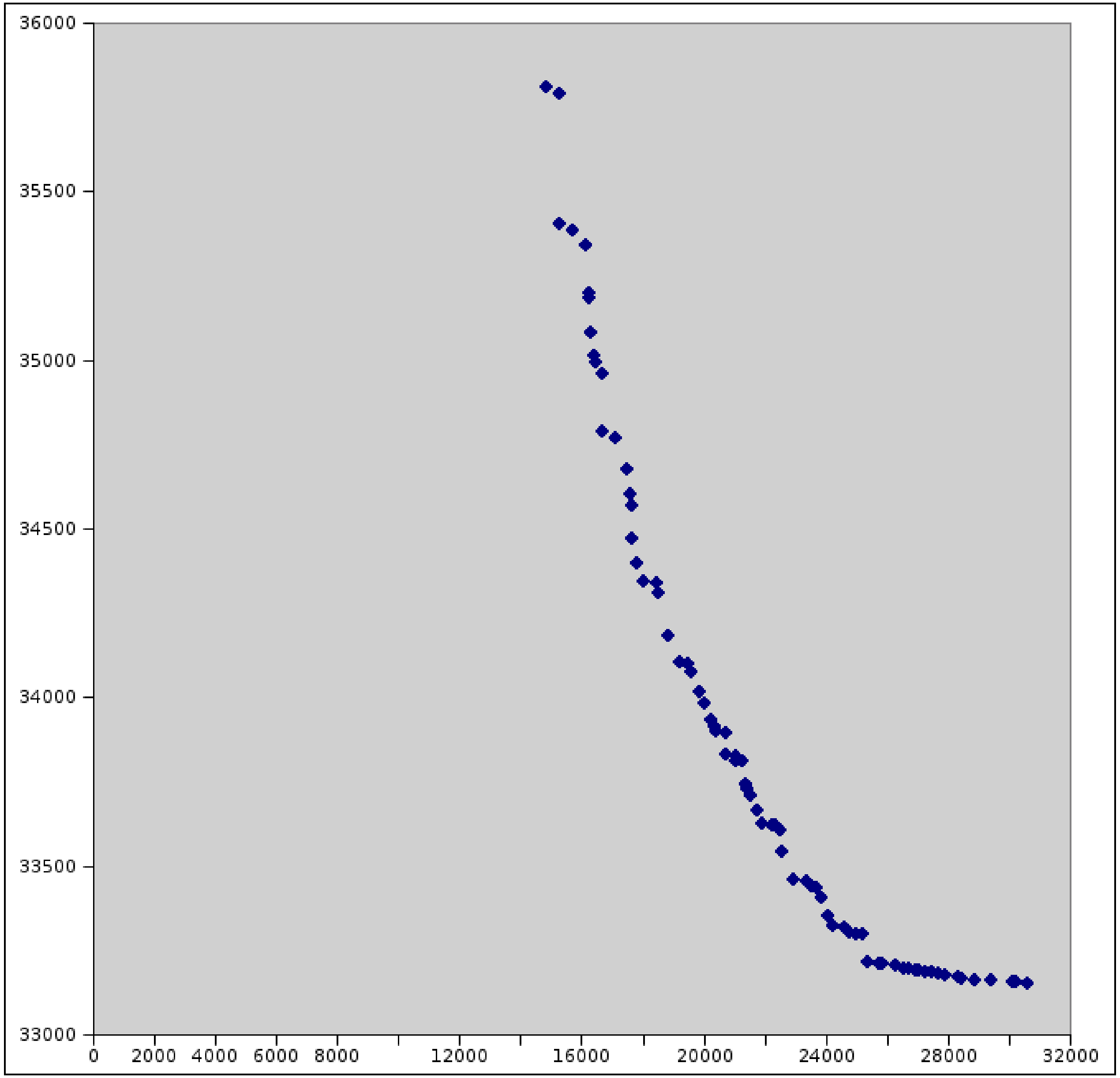


(a) (b) (c) (d)

Application: Pareto Outcomes for a Small CNRP



Application: Pareto Outcomes for a Larger CNRP



Computational Results: Comparing WCN with Bisection Search

Knapsack

	Iterations				Outcomes Found				Max Missed		
	WCN	Δ from WCN			WCN	Δ from WCN					
Size	0	10^{-1}	10^{-2}	10^{-3}	0	10^{-1}	10^{-2}	10^{-3}	10^{-1}	10^{-2}	10^{-3}
10	278	12	300	679	149	-17	0	0	6	0	0
20	364	-1	390	896	192	-22	-2	0	6	1	0
30	324	-43	246	712	167	-25	0	0	4	0	0
40	490	-108	235	898	250	-55	-11	0	5	2	0
50	686	-138	235	1123	348	-69	-9	-1	11	1	1
Totals	2142	-278	1406	4308	1106	-188	-22	-1	11	2	1

CNRP

	Iterations				Outcomes Found				Max Missed		
	WCN	Δ from WCN			WCN	Δ from WCN					
Name	0	10^{-1}	10^{-2}	10^{-3}	0	10^{-1}	10^{-2}	10^{-3}	10^{-1}	10^{-2}	10^{-3}
att48	147	-35	-9	104	74	-18	-15	-4	3	3	1
Totals	2381	-264	724	3794	1207	-135	-13	0	5	1	0

Computational Results: Comparing WCN with ACN

Knapsack

	Iterations				Outcomes Found				Max Missed		
	WCN	Δ from WCN			WCN	Δ from WCN					
Size	0	10^{-2}	10^{-3}	10^{-4}	0	10^{-2}	10^{-3}	10^{-4}	10^{-2}	10^{-3}	10^{-4}
10	278	-4	0	0	149	-2	0	0	1	0	0
20	364	-6	0	0	192	-3	0	0	1	0	0
30	324	-6	0	0	167	-3	0	0	1	0	0
40	490	-24	0	0	250	-12	0	0	1	0	0
50	686	-28	-4	0	348	-24	-2	0	3	2	0
Totals	2142	-70	0	0	1106	-34	-2	0	3	2	0

CNRP

	Iterations				Outcomes Found				Max Missed		
	WCN	Δ from WCN			WCN	Δ from WCN					
Name	0	10^{-2}	10^{-3}	10^{-4}	0	10^{-2}	10^{-3}	10^{-4}	10^{-2}	10^{-3}	10^{-4}
att48	147	-140	-106	-62	74	-70	-53	-31	44	17	8
Totals	2381	-2056	-1012	-34	1207	-1028	-506	-17	18	5	1

Computational Results: Comparing WCN with Hybrid ACN

Knapsack

	Iterations				Outcomes Found				Max Missed		
	WCN	Δ from WCN			WCN	Δ from WCN					
Size	0	10^{-2}	10^{-3}	10^{-4}	0	10^{-2}	10^{-3}	10^{-4}	10^{-2}	10^{-3}	10^{-4}
10	278	-4	0	0	149	-2	0	0	1	0	0
20	364	-6	0	0	192	-3	0	0	1	0	0
30	324	-6	0	0	167	-3	0	0	1	0	0
40	490	-24	0	0	250	-12	0	0	1	0	0
50	686	-28	-4	0	348	-14	-2	0	3	2	0
Totals	2142	-68	-4	0	1106	-34	-2	0	3	2	0

CNRP

	Iterations				Outcomes Found				Max Missed		
	WCN	Δ from WCN			WCN	Δ from WCN					
Name	0	10^{-3}	10^{-4}	10^{-5}	0	10^{-3}	10^{-4}	10^{-5}	10^{-3}	10^{-4}	10^{-5}
att48	147	-106	-62	-6	74	-53	-31	-3	17	8	2
Totals	2381	-1012	-44	-2	1207	-612	-22	-1	5	1	1

Computational Results: Comparing WCN with ACN and Hybrid ACN (CPU Time)

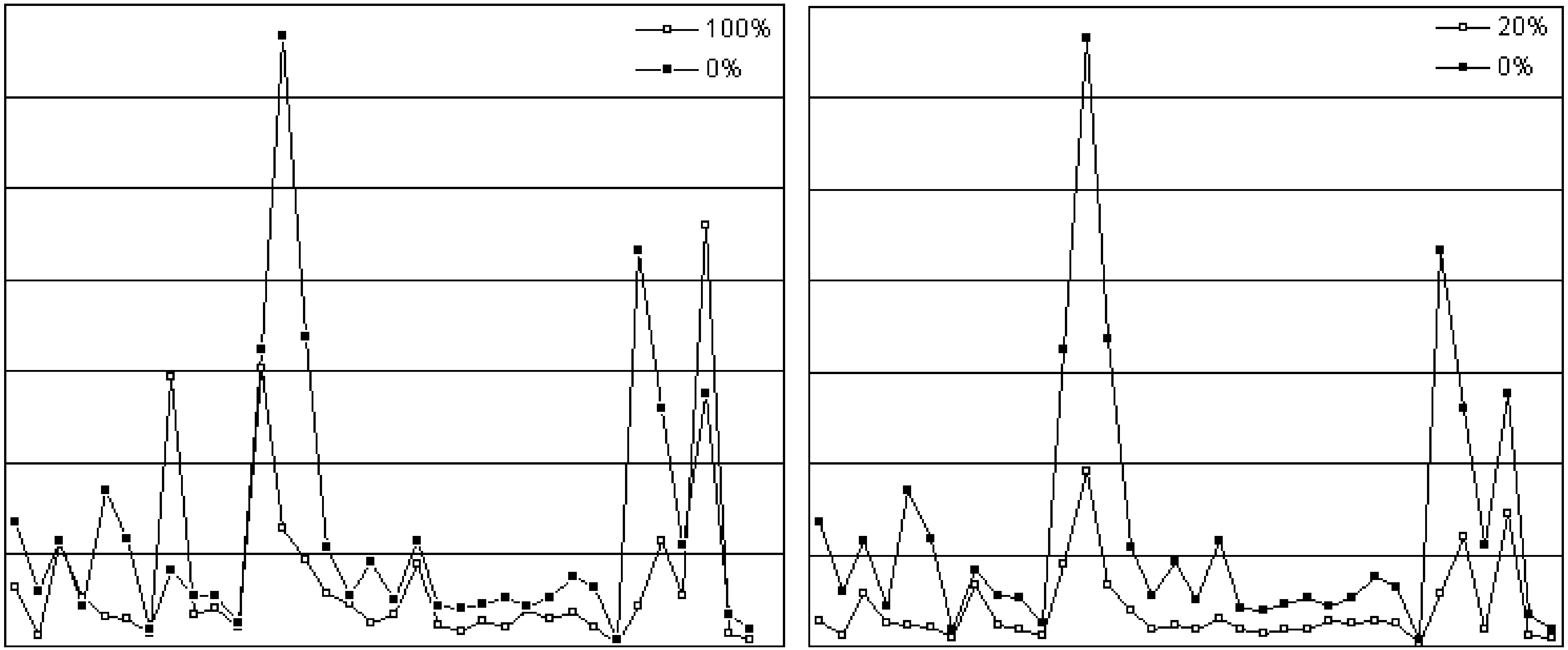
Knapsack

	CPU Time (ACN)				CPU Time (Hybrid)			
	WCN	Δ from WCN			WCN	Δ from WCN		
Size	0	10^{-2}	10^{-3}	10^{-4}	0	10^{-2}	10^{-3}	10^{-4}
10	13.18	0.06	-0.23	-0.10	13.18	0.34	0.12	0.16
20	17.46	-1.33	-0.41	-0.21	17.46	-1.17	0.03	-0.16
30	24.93	-1.28	-0.43	-0.43	24.93	-1.02	-0.11	0.10
40	65.88	-5.69	-1.70	-0.66	24.93	-1.02	-0.11	0.10
50	139.42	-27.18	-3.78	-1.35	65.88	-4.89	-1.09	-0.30
60	260.87	-35.42	-6.55	-2.75	139.42	-13.04	-3.37	-1.17
Totals	260.87	-35.42	-6.55	-2.75	260.87	-19.78	-4.42	-1.37

CNRP

	CPU Time (ACN)				CPU Time (Hybrid)			
	WCN	Δ from WCN			WCN	Δ from WCN		
Name	0	10^{-2}	10^{-3}	10^{-4}	0	10^{-2}	10^{-3}	10^{-4}
att48	83.67	-80.14	-59.83	-28.48	83.67	-59.34	-30.19	-1.12
Totals	8122.36	-7728.51	-5244.54	-1451.37	8122.36	-5481.53	-1531.35	-589.90

Computational Results: Using Warm Starting to Solve CNRP Instances



These are results using SYMPHONY to solve CNRP instances with two different warm starting strategies.

The Next Frontier: Using the Computational Grid

- Enumerating the entire Pareto set may be difficult for hard combinatorial problems.
- This algorithm is, however, naturally parallelizable.
- The order in which the subproblems are solved is not crucial, so there is little need for synchronization.
- Solution of the subproblems themselves can also be parallelized.
- Speedup will depend on the number of subproblems in the queue at any given time.
- Solving the subproblems in different orders may result in different parallel performance.
- We are currently using MW Blackbox to develop a grid-enabled implementation of this algorithm.
- Only the list of breakpoints and solutions generated so far are needed to restart the algorithm.

Conclusion

- Generating the complete set of Pareto outcomes is a challenging computational problem.
- We presented a new algorithm for solving bicriteria mixed-integer programs.
- The algorithm is
  - asymptotically optimal,
  - generates exact breakpoints,
  - has good numerical properties, and
  - can exploits modern solution techniques.
- We have shown how this algorithm is implemented in the SYMPHONY MILP solver framework.
- Future work
  - Improvements to warm starting procedures
  - Parallelization
  - More than two objective

More Information

- SYMPHONY
  - Prepackaged releases can be obtained from `www.BranchAndCut.org`.
  - Up-to-date source is available from `www.coin-or.org`.
  - Available Solvers
    - Generic MILP
    - Traveling Salesman Problem
    - Vehicle Routing Problem
    - Mixed Postman Problem
    - Bicriteria Knapsack Solver
    - Set Partitioning Problem
    - Matching Problem
    - Network Routing
- For references and further details, see *An Improved Algorithm for Biobjective Integer Programming*, to appear in Annals of OR, available from

  `www.lehigh.edu/~tkr2`

- Overviews of multiobjective integer programming
  - Climaco (1997)
  - Ehrgott and Gandibleux (2002)
  - Ehrgott and Wiecek (2005)