Capacitated Vehicle Routing and Some Related Problems

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Outline of Talk

• Introduction and Motivation
• A New Approach
• Complexity and Special Cases
• Valid Inequalities
• Implementation
• Computational Issues and Results
• Future Directions
The Vehicle Routing Problem

The VRP is a combinatorial problem whose ground set is the edges of a graph \( G(V, E) \). Notation:

- \( V \) is the set of customers and the depot (0).
- \( d \) is a vector of the customer demands.
- \( k \) is the number of routes.
- \( C \) is the capacity of a truck.

A feasible solution is composed of:

- a partition \( \{ R_1, \ldots, R_k \} \) of \( V \) such that \( \sum_{j \in R_i} d_j \leq C, \ 1 \leq i \leq k \);
- a permutation \( \sigma_i \) of \( R_i \cup \{0\} \) specifying the order of the customers on route \( i \).
Classical Formulation for the VRP

**IP Formulation:**

\[
\begin{align*}
\sum_{j=1}^{n} x_{0j} &= 2k \\
\sum_{j=1}^{n} x_{ij} &= 2 \quad \forall i \in V \setminus \{0\} \\
\sum_{i \in S, j \notin S} x_{ij} &\geq 2b(S) \quad \forall S \subset V \setminus \{0\}, \ |S| > 1.
\end{align*}
\]

\(b(S)\) = lower bound on the number of trucks required to service \(S\) (normally \(\lceil (\sum_{i \in S} d_i)/C \rceil\)).

If \(C = \sum_{i \in S} d_i\), then we have the Multiple Traveling Salesman Problem.

Alternatively, if the edge costs are all zero, then we have the Bin Packing Problem.
Vehicle Routing and Related Problems

MTSP Polytope

BPP/VRP Polytope

Feasible MTSP/Infeasible BPP
How Hard is the VRP?

- **Test Set**
  - TSPLIB/VRPLIB
  - Augerat’s repository
  - Available at BranchAndCut.org/VRP

- Largest **VRP** instance solved: F-n135-k7
- Smallest **VRP** instance unsolved: B-n50-k8
- Largest **TSP** instance solved: usa13509
- Time to solve B-n50-k8 as an **MTSP**: .1 sec
- Why the gap?
**Standard Approach**

- Standard approaches treat the **VRP** in much the same way as the **TSP**.
  - Most known valid inequalities are generalizations from the **TSP**.
  - Branching rules are also generalizations from the **TSP**.
- However, the TSP does not seem to be the right template.
- It is the packing, not the routing that makes the problem difficult.
What Makes the VRP Difficult?

• It is the intersection of two difficult problems.
  – Traveling Salesman Problem (Routing)
  – Bin Packing Problem (Packing)

• We don’t have an effective, polynomially sized relaxation.
• Current approaches treat it as a routing problem.
• We know very little about the packing aspect.
• We need a different template.
• Idea: Consider flow-based formulations.
Node Routing

- We are given an undirected graph $G = (V, E)$.
  - The nodes represent supply/demand points.
- We consider problems with one supply point (the depot).
- A node routing is a directed subgraph $G'$ of $G$ satisfying the following properties:
  - $G'$ is (weakly) connected.
  - The in-degree of each non-depot node is 1.
Capacitated Node Routing

- A *capacitated node routing* is one in which the demand in each component of $G' \setminus \{0\}$ is $\leq C$.
- Feasible solutions are bin packings.
- This restriction is easily modeled using a flow-based formulation.
- With capacities, we can model the VRP and the Capacitated Spanning Tree Problem (CSTP).
Optimal Node Routing

• Properties of a node routing.
  – It is a spanning arborescence plus (possibly) some edges returning to the depot.
  – There is a unique path from the depot to each demand point.

• We wish to construct a least cost routing.

• Cost Measures
  – Lengths of all edges in $G'$.
  – Length of all paths from the depot.
  – Linear combination of these two.
**IP Formulation**

IP formulation for this routing problem:

$$\text{Min} \sum_{(i,j) \in A} \gamma c_{ij} x_{ij} + \tau c_{ij} f_{ij}$$

s.t.

$$x(\delta(V \setminus \{i\})) = 1 \quad \forall i \in V \setminus \{0\}$$

$$f(\delta(V \setminus \{i\})) - f(\delta(\{i\})) = d_i \quad \forall i \in V \setminus \{0\}$$

$$0 \leq f_{ij} \leq C x_{ij} \quad \forall (i, j) \in A$$

$$x_{ij} \in \{0, 1\} \quad \forall (i, j) \in A$$

where:

- $x_{ij}, x_{ji}$ (fixed-charge variables) indicate whether $\{i, j\}$ is in the routing and its orientation.

- $f_{ij}$ (flow variable) represents demand flow from $i$ to $j$. 

Complexity

- This node routing problem is **NP-complete** even in the uncapacitated case (fixed-charge network flow problem).

- Polynomials solvable special cases.
  - $\tau = 0 \Rightarrow$ Minimum Spanning Tree Problem.
  - $\gamma = 0 \Rightarrow$ Shortest Paths Tree Problem.
  - Note that demands are irrelevant.

- Other special cases.
  - $\tau = 0 \Rightarrow$ Capacitated Spanning Tree Problem.
  - $\tau, \gamma > 0 \Rightarrow$ Cable-Trench Problem.
  - $\tau = 0$ and $x(\delta(\{i\})) = 1 \Rightarrow$ Traveling Salesman Problem.
  - $\tau > 0$ and $x(\delta(\{i\})) = 1 \Rightarrow$ Variable Cost TSP.
  - $x(\delta(V \setminus \{0\})) = x(\delta(\{0\})) = k \Rightarrow$ VRP.
Figure 1: Optimal uncapacitated spanning trees with increasing $\tau/\gamma$ ratios
Figure 2: Uncapacitated vs. capacitated spanning trees ($\tau = 0$)
Connection to Other Models

- There are connections to many well-studied models that may provide better templates.
- The basic model can be seen as an instance of the Fixed-charge Network Flow Problem.
- Removing the upper bounds on the fixed-charge variables yields the Capacitated Network Design Problem.
- We have already mentioned several other related combinatorial models.
- We are looking to make stronger connections among these varied areas of the literature.
Valid Inequalities

- Note that any inequalities valid for the TSP, VRP, or CSTP have counterparts here.
- Many can be strengthened by taking advantage of the directed formulation.
- Fractional Capacity Constraints

\[ \sum_{i \notin S, j \in S} x_{ij} \geq \frac{d(S)}{C}, \ 0 \notin S \]

- Multi-star Inequalities

\[ \sum_{i \notin S, j \in S} x_{ij} \geq \frac{d(S)}{C} + \frac{\sum_{i \notin S, j \in S} x_{ji} d_i}{C}, \ 0 \notin S \]
Valid Inequalities

- Rounded Capacity Constraints

\[ \sum_{i \notin S, \ j \in S} x_{ij} \geq \lceil d(S)/C \rceil \]

- Generalized, framed capacity constraints
- Combs, Hypo-tours, Clique Clusters
- Path-bin inequalities
Flow Linking

- Note that only the edge variables are required to be integral.
- We use the flow variables to force integrality of the edge variables through *flow linking constraints*.
- Flow Linking Constraints

\[
 f_{ij} \leq (C - d_i)x_{ij} \iff x_{ij} \geq \frac{f_{ij}}{C - d_i}
\]

\[
 f_{ij} - \sum_{k \neq j} f_{jk} \leq x_{ij}d_j
\]

- Edge Cuts

\[
x_{ij} + x_{ji} \leq 1
\]
Separation

- The fractional capacity constraints and multi-star inequalities are automatically satisfied.

- Flow linking constraints and edge cuts can be included explicitly or separated in polynomial time.

- Separating rounded capacity constraints is NP-complete, but can be done effectively.

- Heuristic procedures for other classes have not yet been implemented.
**Solver Implementation**

- The implementation uses **SYMPHONY**, a parallel framework for branch, cut, and price (relative of COIN/BCP).

- In **SYMPHONY**, the user supplies:
  - the initial LP relaxation,
  - separation subroutines,
  - feasibility checker, and
  - other optional subroutines.

- **SYMPHONY** handles **everything else**.

- The source code and documentation are available from [www.BranchAndCut.org](http://www.BranchAndCut.org)
Preliminary Computation: Formulation Issues

• The new formulation is polynomial and yields stronger relaxations initially, but there are drawbacks.

• For the VRP, the formulation creates symmetry.

• It also seems to make branching less effective.

• There is a related “undirected” formulation which uses one fixed-charge variable per edge.
  – This formulation is smaller and performs much better for the VRP.
  – For the CSTP and CTP, however, the undirected formulation is extremely weak.
Preliminary Computation: Results So Far

• So far, the presence of the flow variables does not seem to help.

• **Capacitating** the model does increase difficulty significantly.

• Consider relaxations of the **VRP**.
  – The **TSP** is very easy relative to the **VRP**.
  – The **CSTP** is not much easier than the **VRP**.

• Versions of these models with positive variable (flow) costs are extremely difficult.
  – Is this due to the **upper bound** or **lower bound**?
  – The flow linking constraints are important for these models.
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Conclusions and Future Directions

- We have established interesting connections to other well-studied models.
- The TSP does not seem to be the right template to follow.
- We have yet to take full advantage of the information provided by the flow variables.
- Better flow linking seems to be the key.
- We also need some new branching rules.
- The connection to the network design literature needs to be explored.
- We are also considering decomposition-based methods.