Solving Hard Combinatorial Problems: A Research Overview

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Outline of Talk

• Introduction to combinatorial optimization
• Cutting plane methods
• Branch and cut methods
• SYMPHONY
• Future work
Combinatorial Optimization

- A combinatorial optimization problem $CP = (E, \mathcal{F})$ consists of
  - A ground set $E$,
  - A set $\mathcal{F} \subseteq 2^E$ of feasible solutions, and
  - A cost function $c \in \mathbb{Z}^E$ (optional).

- The cost of $S \in \mathcal{F}$ is $c(S) = \sum_{e \in S} c_e$.

- A subproblem is defined by $S \subseteq \mathcal{F}$.

- Problem: Find a least cost member of $\mathcal{F}$.
The Traveling Salesman Problem

The TSP is a combinatorial problem \((E, F)\) whose ground set is the edge set of a graph \(G = (V, E)\).

- \(V\) is the set of customers.
- \(E\) is the set of travel links between the customers.

A feasible solution is a permutation \(\sigma\) of \(V\) specifying the order of the customers.

**IP Formulation:**

\[
\sum_{j=1}^{n} x_{ij} = 2 \quad \forall i \in N^- \\
\sum_{i \in S} x_{ij} \geq 2 \quad \forall S \subset V, \ |S| > 1.
\]

where \(x_{ij}\) is a binary variable indicating \(\sigma(i) = j\).
The Vehicle Routing Problem

In the VRP, we have additional constraints.

- There is a designated depot node (0).
- $d$ is a vector of the customer demands.
- $V^- = V \setminus \{0\}$.
- $k$ is the number of routes.
- $C$ is the capacity of a truck.

A feasible solution is composed of:

- a partition $\{R_1, \ldots, R_k\}$ of $V$ such that $\sum_{j \in R_i} d_j \leq C$, $1 \leq i \leq k$;
- a permutation $\sigma_i$ of $R_i \cup \{0\}$ specifying the order of the customers on route $i$. 
IP Formulation for the VRP

**IP Formulation:**

\[
\sum_{j=1}^{n} x_{0j} = 2k \\
\sum_{j=1}^{n} x_{ij} = 2 \quad \forall i \in V^- \\
\sum_{i \in S} x_{ij} \geq 2b(S) \quad \forall S \subset V^-, \ |S| > 1.
\]

\[b(S) = \text{lower bound on the number of trucks required to service } S \text{ (nominally } \lceil (\sum_{i \in S} d_i)/C \rceil).\]

If \(C = \infty\), then we have the Multiple Traveling Salesman Problem.
How hard are these problems?

I don’t know.
How do we solve these hard problems?

• Try to reduce it to something easier
  - Integer Program $\Rightarrow$ Linear Program
  - Divide and conquer

• Use a bigger hammer
  - Faster processors
  - More memory
  - Parallelism
Integer Programming
Cutting Plane Method

- Basic cutting plane algorithm
  - Relax the integrality constraints.
  - Solve the relaxation. Infeasible ⇒ STOP.
  - If $\hat{x}$ integral ⇒ STOP.
  - Separate $\hat{x}$ from $P$.
  - No cutting planes ⇒ algorithm fails.

- The key is good separation algorithms.
A Separation Algorithm for Side Constraints

- The VRP can be thought of as a side-constrained M-TSP.

- **Key observation**: We can determine in $O(n)$ time whether a particular TSP tour satisfies all the capacity constraints.
VRP Polytope

TSP Polytope

valid inequalities
The Decomposition Algorithm

- Suppose we have two combinatorial problems

\[ CP = (E, \mathcal{F}) \]
\[ CP' = (E, \mathcal{H}) \]

such that \( \mathcal{H} \subseteq \mathcal{F} \).

- Attempt to decompose a given fractional point \( \hat{x} \) into a convex combination of extreme points of \( \mathcal{P} = \text{conv}(\{x^S : S \in \mathcal{F}\}) \).

- If successful, then examine the extreme points in the linear combination to obtain a set of possibly violated inequalities.

- Otherwise, derive a Farkas inequality that separates \( \hat{x} \) from \( \mathcal{P} \).

- This can be implemented using linear programming with column generation.
If the cutting plane approach fails, then we divide and conquer (branch).
Branch and Cut Tree
SYMPHONY

SYMPHONY is a framework for implementing parallel BCP algorithms.

- It was designed specifically to run in a parallel environment.

- User supplies:
  - the initial LP relaxation,
  - separation subroutines,
  - feasibility checker, and
  - other optional subroutines.

- SYMPHONY handles everything else.

- The source code and documentation are available from www.BranchAndCut.org
The Processes of Parallel Branch and Cut

- Master
- GUI
- Tree Manager
- Cut Generator
- LP Solver
- Cut Pool

Diagram showing the processes of parallel branch and cut with arrows indicating the flow of operations between the components.
Current Status and Future Goals

• SYMPHONY
  - First public release early this year.
  - Registered users at 20 universities.
  - The software is mature and contains advanced features available in specialty codes.
  - Research emphasis is on scalability.
    - Parallel scalability (more processors)
    - Data scalability (solving bigger problems)

• Applications
  - Work on the VRP and the decomposition algorithm are continuing.
  - New techniques from the literature to be incorporated into the VRP software.
  - Need out of the box thinking for the next big breakthrough.
  - Looking for other interesting application areas.