Warm Starting for Mixed Integer Linear Programs

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Outline

1. Introduction

2. Algorithms
   - Definitions
   - Implementation

3. Computational Experiments
   - Combinatorial Auctions
   - Primal Heuristics

Motivation

- There are a wide range of applications in which we must repeatedly solve mixed integer linear programs.
- If the instances are only slightly different, can we do better than starting from scratch?
- In principle, warm starting techniques from LP can be generalized to do this job.

Can we make it work in practice?

Applications

Iterative Algorithms

- Bicriteria optimization algorithms
- Primal heuristics (RINS)
- Column generation algorithms for MILP
- Dual decomposition algorithm for stochastic integer programs

Real-time Optimization

- Airline Disaster Recovery
- Stochastic Vehicle Routing
- Combinatorial Auctions

Warm Starting Information

- Many optimization algorithms can be viewed as iterative procedures for satisfying optimality conditions (based on duality).
- These conditions provide a measure of “distance from optimality.”
- Warm starting information is additional input data that allows an algorithm to quickly get “close to optimality.”
- In mixed integer linear programming, the *duality gap* is the usual measure.
- A *starting partition* can quickly reduce the gap.

What is a starting partition and where do we get one?

Optimal Partitions

- Consider the implicit optimality conditions associated with an algorithm branch and bound.
- Let $\mathcal{P}_1, \ldots, \mathcal{P}_s$ be a set of polyhedra whose union contains the feasible set.
- Let B^i be the optimal basis for the LP $\min_{x^i \in \mathcal{P}_i} c^\top x^i$.
- Then the following is a valid lower bound

$$L = \min\{c_{B^i}(B^i)^{-1}b + \gamma_i \mid 1 \le i \le s\}$$

where γ_i is a constant factor associated with the nonbasic variables fixed at nonzero bounds.

- A similar function yields an upper bound.
- We call a partition that yields lower and upper bounds equal is called an *optimal partition*.

Bounding Functions

- The function

$$L(d) = \min\{c_{B^i}(B^i)^{-1}d + \gamma_i \mid 1 \leq i \leq s\}$$

provides a valid lower bound as a function of the right-hand side.

- Here is the corresponding upper bounding function

$$U(c) = \min\{c_{B^i}(B^i)^{-1}b + \beta_i \mid 1 \leq i \leq s, \hat{x}^i \in \Pi\}$$

- These functions can be used for local sensitivity analysis, just as one would do in linear programming.
  - For changes in the right-hand side, the lower bound remains valid.
  - For changes in the objective function, the upper bound remains valid.
  - One can also make other modifications, such as adding variables or constraints.

Data Structures

- To allow resolving from a warm start, we have a data structure for storing warm starts.
- A warm start consists of a snapshot of the search tree, with node descriptions including:
  - lists of active cuts and variables,
  - branching information,
  - warm start information, and
  - current status (candidate, fathomed, etc.).
- The tree is stored in a compact form by storing the node descriptions as differences from the parent.
- Other auxiliary information is also stored, such as the current incumbent.
- A warm start can be saved at any time and then reloaded later.
- The warm starts can also be written to and read from disk.

Warm Starting Procedure

After modifying parameters

- If only parameters have been modified, then the candidate list is recreated and the algorithm proceeds as if left off.
- This allows parameters to be tuned as the algorithm progresses if desired.

After modifying problem data

- After modification, all leaf nodes must be modified appropriately and added to the candidate list.
- After constructing the candidate list, we can continue the algorithm as before.

Pruning the Warm Start

- There are many opportunities for improving the basic scheme.
- For instance, it may not be a good idea to start the warm start from the exact tree produced during a previous solve.
- Any subtree will do.
- Various ad hoc procedures can be used to prune the warm start to produce a smaller tree that may be more effective.
  - First $p\%$ of the nodes produced.
  - All nodes above level $p * \text{max}$, $0 \leq p \leq 1$.
  - Top k levels.

SYMPHONY: Support for Warm Starting

Currently supported

- Change to objective function (no reduced cost fixing during generation of warm start).
- Change to right hand side (cuts are discarded when resolving).
- Changes to variable bounds.
- Addition of columns.

Coming soon

- Addition of constraints (easy).
- Changes to right hand side without discarding cuts (not so easy).
- Changes to objective function with reduced cost fixing (not so easy).

Basic Sensitivity Analysis

SYMPHONY will calculate bounds after changing the objective or right-hand side vectors.

```
int main(int argc, char **argv)
{
   OsiSymSolverInterface si;
   si.parseCommandLine(argc, argv);
   si.loadProblem();
   si.setSymParam(OsiSymSensitivityAnalysis,
                  true);
   si.initialSolve();
   int ind[2];
   double val[2];
   ind[0] = 4;   val[0] = 7000;
   ind[1] = 7;   val[1] = 6000;
   lb = si.getLbForNewRhs(2, ind, val);
}
```

Warm Starting Example (Parameter Modification)

The following example shows a simple use of warm starting to create a dynamic algorithm.

```
int main(int argc, char **argv)
{
   OsiSymSolverInterface si;
   si.parseCommandLine(argc, argv);
   si.loadProblem();
   si.setSymParam(OsiSymFindFirstFeasible,true);
   si.setSymParam(OsiSymSearchStrategy,
                  DEPTH_FIRST_SEARCH);
   si.initialSolve();
   si.setSymParam(OsiSymFindFirstFeasible,false);
   si.setSymParam(OsiSymSearchStrategy,
                  BEST_FIRST_SEARCH);
   si.resolve();
}
```

Warm Starting Example (Problem Modification)

This example shows how to warm start after problem modification.

```
int main(int argc, char **argv)
{
   OsiSymSolverInterface si;
   CoinWarmStart ws;
   si.parseCommandLine(argc, argv);
   si.loadProblem();
   si.setSymParam(OsiSymNodeLimit, 100);
   si.initialSolve();
   ws = si.getWarmStart();
   si.resolve();
   si.setObjCoeff(0, 1);
   si.setObjCoeff(200, 150);
   si.setWarmStart(ws);
   si.resolve();
}
```

Iterative Combinatorial Auctions

- Bidders create desired packages of items and submit bids on them.
- In each round, the provisional set of winning packages are determined. by maximizing revenue for round t.
- Constraints ensure that each item is awarded at most once.
- Additional constraints may require that a bidder win no more than one package.

The Winner Determination Problem

Set Packing Formulation

$$\max \sum_{j \in S^t} c_j x_j$$

$$\text{s.t.} \sum_{j \in S^t} a_{ij} x_j \leq 1, \text{ for all } i \in S$$

$$x_j \in \{0, 1\} \text{ for all } j \in S^t,$$

$S^t =$ set of considered packages in round t,
$c_j =$ the bid amount of package j,
$S =$ the set of items being auctioned,

$$a_{ij} = \begin{cases} 1, & \text{if item } i \text{ is in package } j; \\ 0, & \text{otherwise.} \end{cases}$$

$$x_j = \begin{cases} 1, & \text{if package } j \text{ is a winning package;} \\ 0, & \text{otherwise.} \end{cases}$$

Set Partitioning Formulation

Alternatively, we can formulate the WDP as a set partitioning problem.

$$\max \sum_{j \in S^t} c_j x_j$$

$$\text{s.t.} \sum_{j \in S^t} a_{ij} x_j = 1, \text{ for all } i \in S$$

$$x_j \in \{0, 1\} \text{ for all } j \in S^t,$$

- In each round t, all items are awarded.
- This formulation is used under the assumption of zero disposal value.

Using Warm Starting

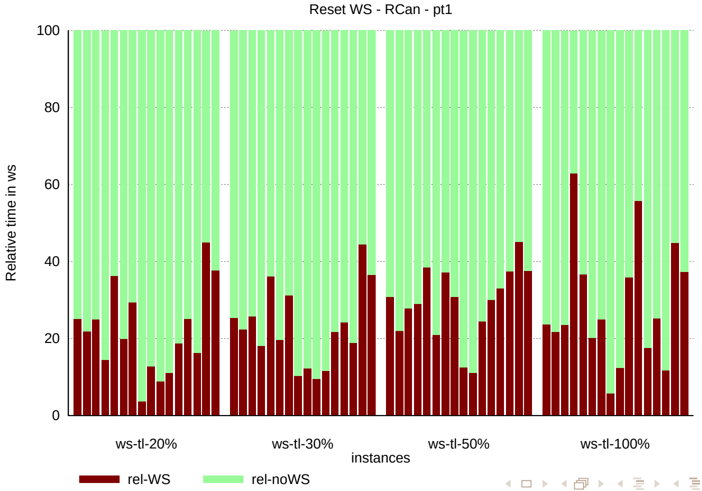
- After each round, the bidders receive feedback and nonwinning bidders can
  - increase their bids for the packages already in the auction, or
  - create bids for new packages.
- In the next round, we need to solve a modified WDP with
  - updated objective coefficients,
  - new columns, or
  - both

Computational Experiments

- We use the cuts generated by SYMPHONY's SPP+CUTS package
  - star cliques
  - odd holes
  - odd antiholes
- These cuts remain valid from round to round.
- We lift the star cliques (greedily) when new columns added
- Two test cases for Set Packing and Set Partitioning cases
  1. Reset WS after each iteration.
  2. Reset WS dynamically based on
     - total #of columns added
     - change in solution times
- # of bidders (5-20) - # of items(3-18)

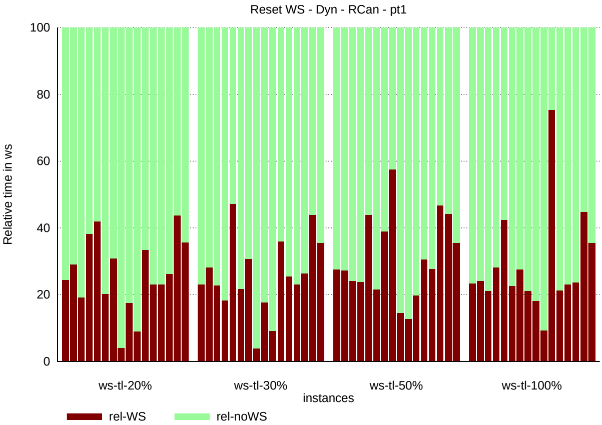
WDP: Set Packing Formulation

Reset WS every time:



WDP: Set Packing Formulation

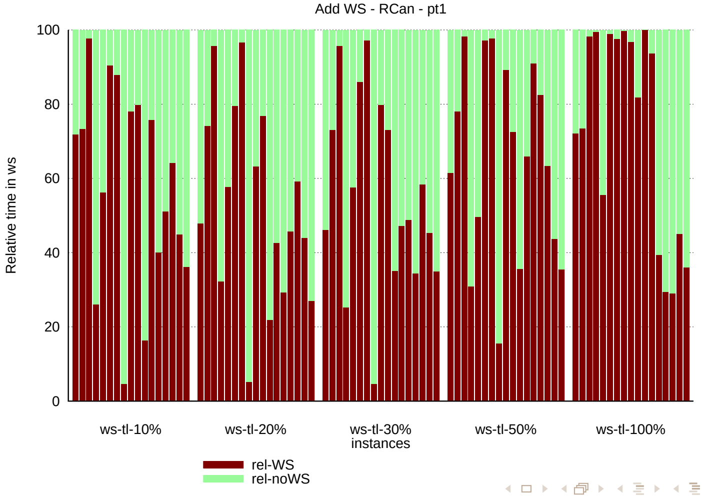
Reset WS dynamically:



Max reset %:
WS-20% $\longrightarrow$ 8.8% (59/672)
WS-30% $\longrightarrow$ 8.7% (58/668)
WS-50% $\longrightarrow$ 13.2% (36/272)
WS-100% $\longrightarrow$ 9.2% (25/272)

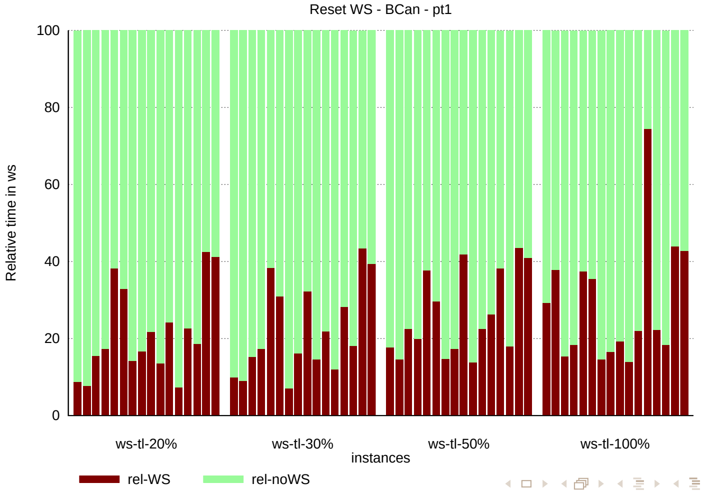
WDP: Set Packing Formulation

Never reset WS:



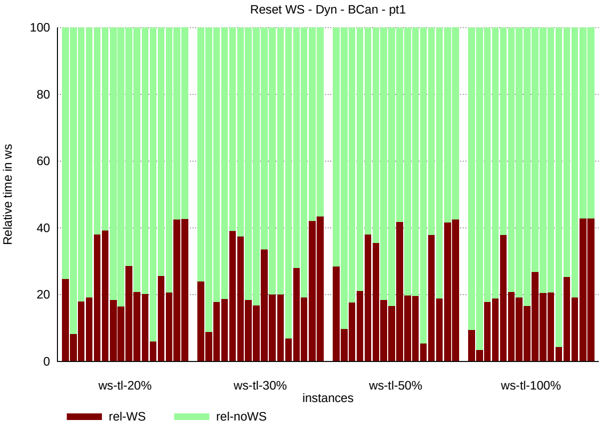
WDP: Set Partitioning Formulation

Reset WS every time:



WDP: Set Partitioning Formulation

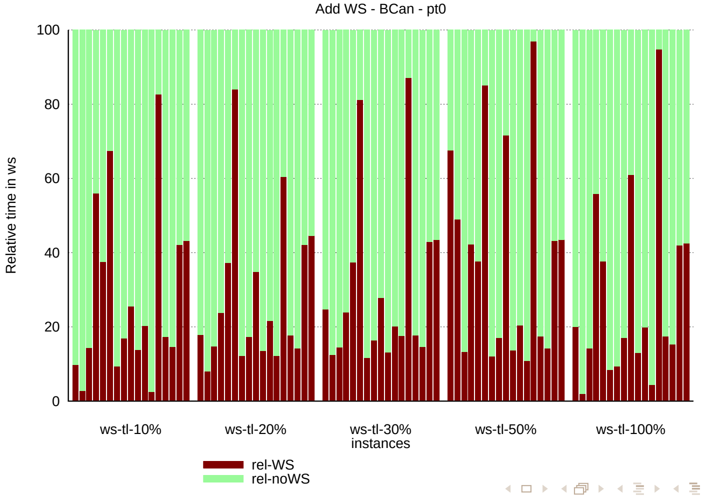
Reset WS dynamically:



Max reset %:
20% ⟶ 9.3% (27/290)
30% ⟶ 8.9% (44/497)
50% ⟶ 11.4% (33/290)
100% ⟶ 8.9% (26/289)

WDP: Set Partitioning Formulation

Never reset the warm start:



Relaxation Induced Neighborhood Search

- An improvement heuristic by Dana, Rothberg, Pape [2005]
- Suppose x^* is an incumbent solution, x^{LP} is a solution of the LP relaxation at a particular node.

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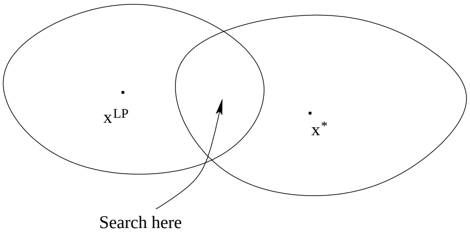
- Explore some common neighborhood of x^{LP} and x^* for a better feasible solution.



Relaxation Induced Neighborhood Search

- An improvement heuristic by Dana, Rothberg, Pape [2005]
- Suppose x^* is an incumbent solution, x^{LP} is a solution of the LP relaxation at a particular node.
- Explore some common neighborhood of x^{LP} and x^* for a better feasible solution.
- Neighborhood is defined by fixing those x_i for which $x_i^{LP} = x_i^*$
- It is explored by calling a MIP solver

$$\begin{aligned} x^{LP} &= (0, 0, 1, 1, 0, 0.5, 0.2, 1, 0.2, 0) \\ x^* &= (1, 0, 0, 1, 1, 0.0, 1.0, 1, 1.0, 1) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} &x \in P(\text{original problem}) \\ &x_1 = 0 \\ &x_3 = 1 \\ &x_7 = 1 \end{aligned}$$

[SOLVE AS A MIP]

Warm Starting in RINS

Motivation

- If RINS is being called repeatedly, the sub-MILPs being solved will be similar.
- Perhaps warm starting can help.

Experimental setup

- We initially save the root node (LP+Cuts) as a warm start.
- Each time RINS is called:
  - Load the current warm start environment.
  - Fix x_i if $x_i^{LP} = x_i^*$ by changing bounds.
  - Solve with time limit.

Computational experiments

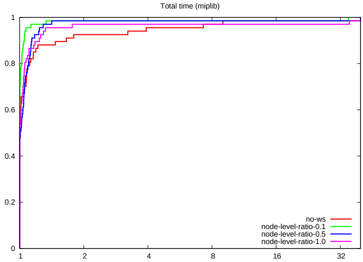
- All mixed binary instances in miplib3(50), miplib2003(45), mittleman-unibo-instances(31).
- Compared 6 runs: Default RINS, warm-starting with node-level-ratio 0.1, 0.2, 0.5, 0.7, 1.0
- Total time limit = 2hrs. (4 hrs for mittleman-unibo)
- Time limit for each RINS call = 100s (300s for mittleman-unibo)

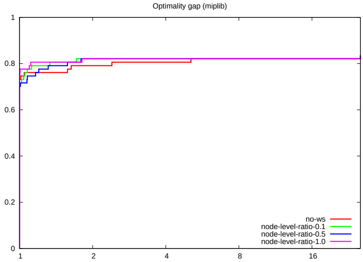
Average behavior over all instances

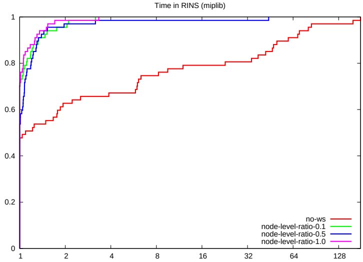
	Def	nl-ratio=0.1	0.2	0.5	0.7	1.0
Avg. $\frac{\text{Time-per-call-Def}}{\text{Time-per-call}}$	1.0	2.53	2.49	2.46	2.48	2.59
Total # heuristic solutions	158	199	190	184	215	210

How does the performance vary over the instances?

MIPLIB Instances: Performance Profiles

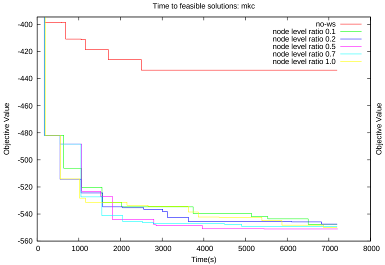


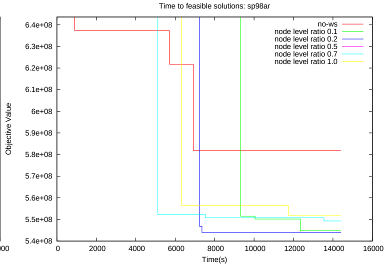


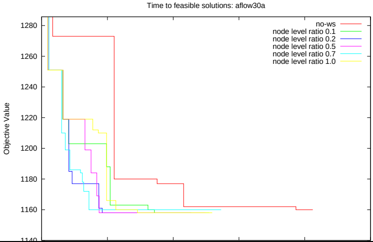


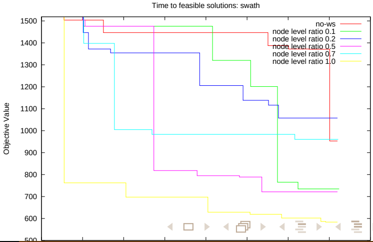
- Time spent in RINS is reduced dramatically
- Minor improvements in running time and optimality gap
- Not unexpected as only upper-bounds are affected

Results: Particular instances









Conclusions and Future Work

- Combinatorial Auctions
  - For combinatorial auctions, using warm starting sped up computations consistently.
  - It was necessary to start from scratch periodically in order to avoid a loss of efficiency.
- RINS
  - Using warm start improved time spent by a factor of 2-3 times for medium and (some) large instances.
  - Overall impact was low, however.
  - For very large instances, carrying over a single warm-start environment does not seem to be a good idea.
  - Starting from scratch if several calls to heuristic are unsuccessful might be a good idea.
  - Not having cuts and reduced cost fixing is a handicap.
- In both cases, it was difficult to know how to prune the warm start intelligently.
- The bottom line: when solving modified instances of these classes of MILPs, warm starting does seem to be a good idea.