In Search of Optimal Disjunctions

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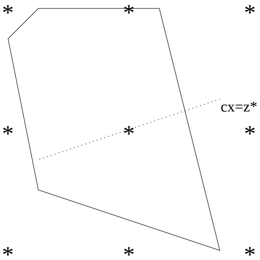
Solving MILPs with Branch and Bound

$$\text{MILP:} \min_{x \in S} cx$$

$$S: \begin{array}{rcl} Ax & \geq & b \\ x & \in & \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{array}$$



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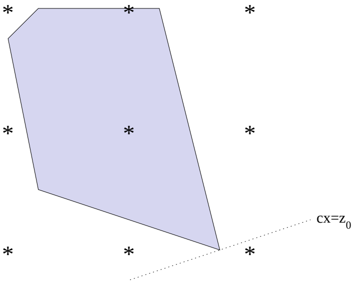

LB = z_0

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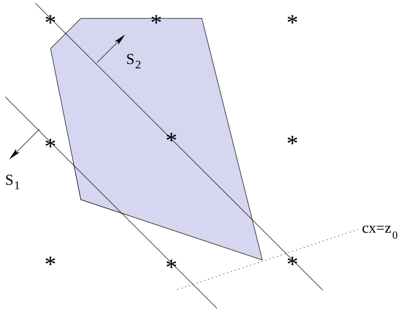

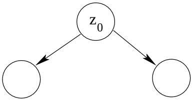

$$\text{LB} = z_0,\ x^* \in S_1 \cup S_2$$

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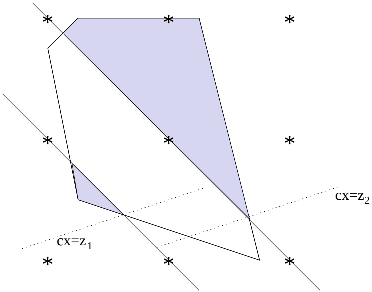

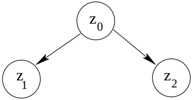

$$\text{LB} = \min(z_1, z_2)$$

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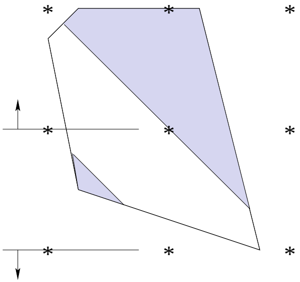

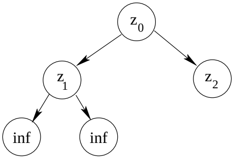

LB = z_2

Solving MILPs with Branch and Bound

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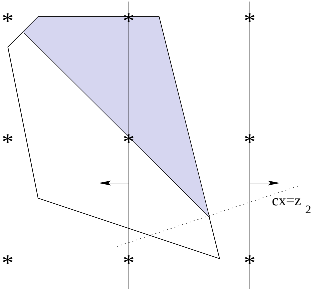

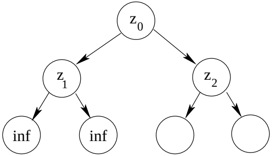

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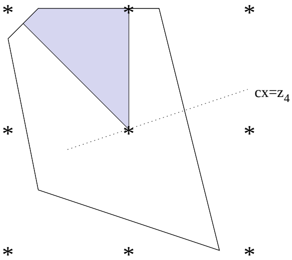
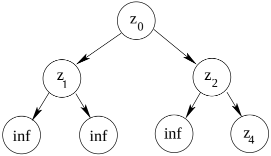




$$\text{LB} = z_4 = \text{UB}$$

Disjunctions

What is the underlying principle?

Definition

A *(linear) disjunction* is an operator on a countable set of systems of inequalities that results in TRUE if and only if at least one of the systems has a feasible solution.

- We write a disjunction over a given set $\mathcal{S}$ as

$$\bigvee_{h \in \mathcal{Q}} A^h x \geq b^h, \ x \in \mathcal{S} \tag{1}$$

  where $A^h \in \mathbb{Q}^{m_h \times n}$, $b^h \in \mathbb{Q}^{m_h}$, $n \in \mathbb{N}$, $m_h \in \mathbb{N}$, $h \in \mathcal{Q}$.
- The disjunction is TRUE if and only if there exist $\tilde{x} \in \mathcal{S}, h \in \mathcal{Q}$ such that $A^h \tilde{x} \geq b^h$.

Valid Disjunctions

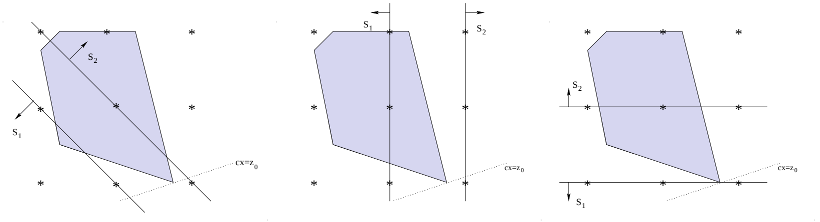
Definition

A disjunction is *valid* for a set $\mathcal{S}$ if it is TRUE for all members of $\mathcal{S}$

Almost all algorithms for solving MILPs can be thought of in terms of the following loop:

1. Solve a linear relaxation to obtain $\hat{x} \in \mathbb{R}^n$.
2. Determine whether there is a valid disjunction for the feasble region that is FALSE for $\hat{x}$.
   - If none, then $\hat{x}$ is optimal.
   - Otherwise, impose the disjunction and return to Step 1.

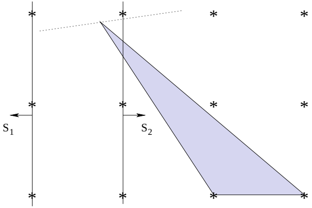
The Branching Decision



- Any subsets S_1, S_2 such that $S \subseteq (\mathcal{P} \cup S_1) \cup (\mathcal{P} \cap S_2)$ could be used to partition.
- We are interested in partitions arising from binary disjunctions.
- There are an infinite number of valid such disjunctions in general.
- A *good* partition is one that can be solved *easily*.
- How should we select such a partition? How should we divide?

The Branching Decision

- Most commonly used branching rule is $x_i \leq \pi_0 \vee x_i \geq \pi_0 + 1$ for an $i \in \{1, \ldots, d\}$.
- e.g. $S_1 = \{x | x_1 \leq 0\}, S_2 = \{x | x_1 \geq 1\}$.
- This is called Variable Disjunction. Also denoted as: $x_1 \leq 0 \vee x_1 \geq 1$.



A variable disjunction

The Branching Decision

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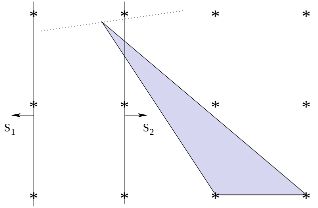
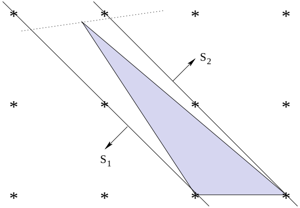
- e.g. $S_1 = \{x | x_1 \leq 0\}, S_2 = \{x | x_1 \geq 1\}$.
- This is called Variable Disjunction. Also denoted as: $x_1 \leq 0 \vee x_1 \geq 1$.
- Disjunctions like $x_1 + x_2 \leq 4 \vee x_1 + x_2 \geq 5$ are also valid.
- A General Disjunction is of the form $\pi x \leq \pi_0 \vee \pi x \geq \pi_0 + 1$, where $(\pi, \pi_0) \in \mathbb{Z}^d \times 0^{n-d} \times \mathbb{Z}$.



A variable disjunction



A general disjunction

Previous Work

Branching on Variables:

- Finding the minimum size tree is $\mathcal{NP}-$Hard. (Liberatore, 2000)
- Heuristics for improving bounds: Strong branching, Pseudo-cost branching, Reliability branching (Experiments by Linderoth and Savelsbergh, 1999, Achterberg et al., 2005)
- Heuristics for feasible solutions (Patel and Chinneck, 2007)
- Heuristics for conflict analysis and resolution (Chvátal, 1997, Achterberg, 2007)
- Heuristics based on thin directions (Derpich and Vera, 2006)

Branching on General Disjunctions:

- Trees polynomial in size, when dimension is fixed (Lenstra 1983, other works by Aardal, Lovasz, Lenstra, Pataki).
- Greedy heuristic by Owen and Mehrotra, 2001.
- Local branching heuristic for feasibility (Fischetti and Lodi, 2003).
- Heuristics based on Gomory Cuts (Karamanov and Cornuéjols, 2007, Cornuéjols et. al., 2008).
- SOS-1 and SOS-2 (Beale, 1970).

Branching on General Disjunctions

$$\text{MILP:} \quad \begin{aligned} z^* &= \min cx \\ \text{s.t. } Ax &\geq b \\ x &\in \mathbb{Z}^n \end{aligned}$$

$$\text{LP:} \quad \begin{aligned} z^*_{LP} &= \min cx \\ \text{s.t. } Ax &\geq b \\ x &\in \mathbb{R}^n \end{aligned}$$

$$\begin{aligned} z^*_L = \min cx & \\ \text{s.t. } Ax &\geq b \\ \pi x &\leq \pi_0 \\ x &\in \mathbb{R}^n \end{aligned} \qquad \begin{aligned} z^*_R = \min cx & \\ \text{s.t. } Ax &\geq b \\ \pi x &\geq \pi_0 + 1 \\ x &\in \mathbb{R}^n \end{aligned}$$

New lower bound, $z_l = \min\{z^*_L, z^*_R\}$

Objective

Find $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $z_l = \min\{z^*_L, z^*_R\}$ is maximized

Complexity

Optimization Problem

Find $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $z_l = \min\{z_L^*, z_R^*\}$ is maximized.

- For branching on variables, this can be solved in polynomial-time.
- Is there a similar result when considering general disjunctions?
- When $(\pi, \pi_0) \in \{0, 1\}^{n+1}$?

Decision Problem

Given a MILP and K, does there exist $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $z_l = \min\{z_L^*, z_R^*\} > K$?

Disjunctive Proof of Infeasibility

Given a MILP, does there exist $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that the LP relaxation of each branch is infeasible?

Complexity Results

Disjunctive Proof of Infeasibility

Given a MILP, does there exist $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that LP relaxation of each branch is infeasible.

Above problem is:

- $\mathcal{NP}-$complete in general (Reduction from Number Partitioning Problem)
- $\mathcal{NP}-$complete when $\pi \in \{0, 1\}^n$
- $\mathcal{NP}-$complete when $\pi \in \{-1, 0, 1\}^n$
- $\mathcal{NP}-$complete when $(\pi, \pi_0) \in \{0, 1\}^{n+1}$
- $\mathcal{NP}-$complete when $x \in \{0, 1\}^n$ and either of above four conditions hold (Reduction from 1-IN-3SAT)

Optimization Problem

Find $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $z_l = \min\{z_L^*, z_R^*\}$ is maximized.
$\mathcal{NP}-$hard

Problem formulation: Maximum bound improvement

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z_L^* = \min\ & cx \\
\text{s.t. } Ax &\geq b \\
\pi x &\leq \pi_0 \\
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\qquad
\begin{aligned}
z_R^* = \min\ & cx \\
\text{s.t. } Ax &\geq b \\
\pi x &\geq \pi_0 + 1 \\
x &\in \mathbb{R}^n
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Decision Problem

Given a MILP and K, does there exist $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $z_l = \min\{z_L^*, z_R^*\} > K$?

Suppose $(\hat{\pi}, \hat{\pi}_0)$ is one such disjunction. Then the following systems must be infeasible

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\begin{aligned}
Ax &\geq b \\
cx &\leq K \\
\hat{\pi} x &\leq \hat{\pi}_0 \\
x &\in \mathbb{R}^n
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How to deal with bilinear terms $\hat{\pi} x$?

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$$\begin{array}{rcl} z_1^* & = & \min \hat{\pi}x \\ Ax & \geq & b \\ cx & \leq & K \\ x & \in & \mathbb{R}^n \end{array} \quad \& \quad \begin{array}{rcl} z_2^* & = & \min -\hat{\pi}x \\ Ax & \geq & b \\ cx & \leq & K \\ x & \in & \mathbb{R}^n \end{array}$$

must have $z_1^* > \hat{\pi}_0$, $z_2^* > -\hat{\pi}_0 - 1$

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$$\begin{array}{rcl} z_1^* = \max pb & - & s_L K \\ pA - s_L c & = & -\hat{\pi} \\ p, s_L & \geq & 0 \end{array} \quad \& \quad \begin{array}{rcl} z_2^* = \max qb & - & s_R K \\ qA - s_R c & = & -\hat{\pi} \\ q, s_R & \geq & 0 \end{array}$$

(as above)

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(as above)

$$\begin{array}{rcl} pb - s_L K & > & \hat{\pi}_0 \\ pA - s_L c & = & -\hat{\pi} \\ p, s_L & \geq & 0 \end{array} \quad \& \quad \begin{array}{rcl} qb - s_R K & > & -\hat{\pi}_0 - 1 \\ qA - s_R c & = & -\hat{\pi} \\ q, s_R & \geq & 0 \end{array}$$

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Problem formulation: Maximum bound improvement

If $(\hat{\pi}, \hat{\pi}_0)$ is the required disjunction,

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Problem formulation: Maximum bound improvement

If $(\hat{\pi}, \hat{\pi}_0)$ is the required disjunction,

$$\begin{aligned} Ax &\geq b \\ cx &\leq K \\ \hat{\pi}x &\leq \hat{\pi}_0 \\ x &\in \mathbb{R}^n \end{aligned} \quad \& \quad \begin{aligned} Ax &\geq b \\ cx &\leq K \\ \hat{\pi}x &\geq \hat{\pi}_0 + 1 \\ x &\in \mathbb{R}^n \end{aligned} \quad \text{must be infeasible}$$

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$(\hat{\pi}, \hat{\pi}_0)$ is the required disjunction $\iff$ it satisfies the last MILP.

Problem formulation: Maximum bound improvement

$$
\begin{aligned}
pb - s_L K &> \pi_0 \\
pA - s_L c &= -\pi \\
qb - s_R K &> -\pi_0 - 1 \\
qA - s_R c &= -\pi \\
q, s_R, p, s_L &\geq 0 \\
(\pi, \pi_0) &\in \mathbb{Z}^n
\end{aligned}
$$

- $2n + 2$ constraints, $2m + n + 3$ variables.
- Tells us if we can increase the bound to K by branching at the current node.
- Does not tell us the maximum such value of K.
- We do a binary search over a range of K and solve the above MILP in each iteration.

Problem formulation: Thin Directions

Similar approach has been used before:

- Cut generating LPs (Balas)
- Separation of a given point from the the *Split Closure* (Caprara, 2003, Balas 2007)
- Finding thinnest directions?

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- Finding thinnest directions?

Thin Directions

- Thickness of the LP relaxation polytope $\mathcal{P}$ along a direction $\hat{\pi}$ is:

$$\max \hat{\pi} y - \hat{\pi} x, \quad x, y \in \mathcal{P}$$

- The width of $\mathcal{P}$, $w(\mathcal{P})$, is minimum such width over all $\hat{\pi}$:

$$w(Q) = \min_{\pi} \max_{x,y \in \mathcal{P}} (\pi y - \pi x) \ \ s.t. \ \ \pi \in \mathbb{Z}^d \times \{0\}^{n-d}, \pi \neq \mathbf{0}.$$

Problem formulation: Thin Directions

$$w(Q) = \min_{\pi} \max_{x,y \in \mathcal{P}} (\pi y - \pi x) \;\; s.t. \;\; \pi \in \mathbb{Z}^d \times \{0\}^{n-d}, \pi \neq \mathbf{0}.$$

For a fixed $(\hat{\pi}, \hat{\pi}_0)$, inner LP is:

$$\begin{aligned} \max \hat{\pi} y - \hat{\pi} x \\ \text{s.t. } Ax \geq b \\ Ay \geq b \end{aligned}$$

Dual of inner LP is:

$$\begin{aligned} \min -qb - pb \\ \text{s.t. } pA - \hat{\pi} = 0 \\ qA + \hat{\pi} = 0 \\ p, q \geq 0 \end{aligned}$$

Over all π:

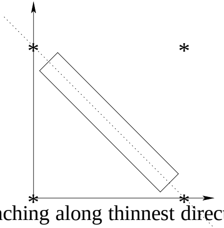
$$\begin{aligned} \min -qb - pb \\ pA - \pi = 0 \\ qA + \pi = 0 \\ p, q \geq 0 \\ \pi \in \mathbb{Z}^n \times \{0\}^{n-d}, \pi \neq \mathbf{0}. \end{aligned}$$

Optional constraint:

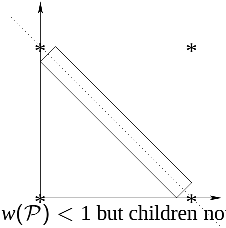
$$\pi_0 < \pi x^* < \pi_0 + 1, \quad \pi_0 \in \mathbb{Z}$$

More on Thin Directions

- Problem of finding the thinnest direction is $NP-$hard. Remains hard even if the given LP polytope is a simplex (Sebö, 1999).
- Branching on thinnest direction does not necessarily result in the smallest tree (Examples by Krishnamoorthy, 2008).
- Branching on the thinnest direction need not make the child nodes infeasible.
- If a general branching disjunction makes the children infeasible then $w(\mathcal{P}) < 1$. Converse is not true.



Branching along thinnest direction gives a feasible LP.



$w(\mathcal{P}) < 1$ but children not infeasible for any disjunction.

Computational Experiments

- Objective: Study the effect of branching on *best* general hyperplanes on the size of branch-and-bound tree.
- Ignore time taken to solve.
- Used ILOG CPLEX-10.2 MILP solver.
- Compared against strong branching (our own callback).
- Simple branch-and-bound (no cuts, preprocessing, heuristics).
- Provided best known upper bound to the solver.
- To perform experiments in reasonable time:
  - Imposed a limit of $-M \le \pi_i \le M$ (M is a fixed parameter).
  - Imposed a constraint of $|\sum_i \pi_i| \le k$ (k is a fixed parameter).
  - Each MILP solved for finding a branching hyperplane had time limit t seconds (t is a fixed parameter).
  - Limit time on branching on a node to $8t$ seconds
  - If more than 18 hours spent on branching, then switch to strong branching on variables.
  - Total time limit of 20 hours for each instance.

Effect of Branching with General Disjunctions

Instance	N_1	N_5	r_5	N_{10}	r_{10}	N_{15}	r_{15}	N_{20}	r_{20}
10teams	115	28	4.11	18	6.39	12	9.58	12	9.58
aflow30a	36634	19485	1.88	20388	1.8	24112	1.52	20271	1.81
bell3a	16387	8771	1.87	588	27.87	259	63.27	259	63.27
blend2	304	231	1.32	188	1.62	165	1.84	209	1.45
egout	2246	554	4.05	572	3.93	676	3.32	558	4.03
fiber	18412	7612	2.42	3039	6.06	3358	5.48	3324	5.54
flugpl	394	6	65.67	10	39.4	6	65.67	6	65.67
gen	100	100	1	100	1	100	1	100	1
gesa2	33526	21664	1.55	21849	1.53	21849	1.53	21778	1.54
gesa2_o	98550	24435	4.03	24661	4	24661	4	24661	4
gt2	340	10	34	12	28.33	10	34	12	28.33
harp2	432010	174656	2.47	183306	2.36	174454	2.48	179130	2.41
khb05250	738	594	1.24	588	1.26	614	1.2	618	1.19
l152lav	60	32	1.88	28	2.14	34	1.76	30	2
lseu	4058	226	17.96	78	52.03	58	69.97	58	69.97
mod008	2840	296	9.59	102	27.84	68	41.76	52	54.62
neos6	5989	2131	2.81	2131	2.81	2131	2.81	2131	2.81
nug08	14	4	3.5	6	2.33	6	2.33	5	2.8
nw04	30	16	1.88	12	2.5	12	2.5	12	2.5
p0548	1050	466	2.25	566	1.86	565	1.86	565	1.86
pp08aCUTS	1301300	147271	8.84	166943	7.79	168905	7.7	231527	5.62
qnet1	42	24	1.75	20	2.1	22	1.91	18	2.33
qnet1_o	154	94	1.64	77	2	80	1.93	92	1.67
ran10x26	68449	23309	2.94	24716	2.77	23704	2.89	21520	3.18
ran12x21	494558	219967	2.25	208948	2.37	225980	2.19	212910	2.32
ran13x13	124716	74699	1.67	57825	2.16	66008	1.89	58789	2.12
rout	219322	65201	3.36	61806	3.55	61226	3.58	57673	3.8
stein45	31086	21238	1.46	20594	1.51	20601	1.51	20601	1.51
vpm1	263111	145	1814.56	32	8222.22	20	13155.55	5929	44.38
vpm2	273994	77504	3.54	67014	4.09	69515	3.94	73687	3.72

Valid Inequalities from Disjunctions

- For the rest of the talk, we'll focus on the case of a pure integer program

$$\begin{aligned} z_{IP} &= \max\{cx \mid x \in S\}, \\ S &= \{x \in \mathbb{Z}^n_+ \mid Ax \leq b\}. \end{aligned}$$

- Valid inequalities for $\mathrm{conv}(S)$ can also be generated from any disjunction.
- Let $\mathcal{P}_i = \{x \in \mathbb{R}^n_+ \mid A^i x \leq b^i\}$ for $i = 1, \dots, k$ be such that $S \subseteq \cup_{i=1}^k \mathcal{P}_i$.
- Then inequalities valid for $\cup_{i=1}^k \mathcal{P}_i$ are also valid for $\mathrm{conv}(S)$.

The Disjunctive Principle

Valid inequalities based on disjunctions can be generated by applying the following simple principle:

Proposition

If $\sum_{j=1}^{n} \pi_j^1 \leq \pi_0^1$ is valid for $S_1 \subseteq \mathbb{R}_+^n$ and $\sum_{j=1}^{n} \pi_j^2 \leq \pi_0^2$ is valid for $S_2 \subseteq \mathbb{R}_+^n$, then

$$\sum_{j=1}^{n} \min(\pi_j^1, \pi_j^2)x \leq \max(\pi_0^1, \pi_0^1)$$

is valid for $x \in S_1 \cup S_2$.

Valid Inequalities for an Explicitly Described Polyhedron

- Consider the feasible region of the LP relaxation $\mathcal{P} = \{x \in \mathbb{R}^n_+ \mid Ax \leq b\}$.
- Valid inequalities for $\mathcal{P}$ can be obtained by taking nonnegative linear combinations of the rows of (A, b).
- Except for one pathological case[1], all valid inequalities for $\mathcal{P}$ are either equivalent to or dominated by an inequality of the form

$$uAx \leq ub, u \in \mathbb{R}^m_+.$$

- To avoid the pathological case, we may assume that A contains explicit upper bounds on the variables.

[1] The pathological case occurs when one or more variables have no explicit upper bound *and* both the primal and dual problems are infeasible.

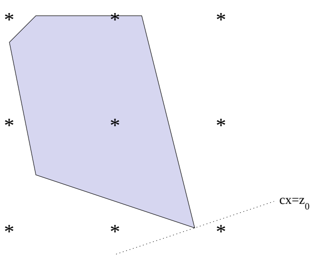
Valid Inequalities from Disjunctions (Algebraic View)

All valid inequalities for the union of two polyhedra can be obtained in this way.

Proposition

If $\mathcal{P}^i = \{x \in \mathbb{R}^n_+ \mid A^i x \leq b^i\}$ for $i = 1, 2$ are nonempty polyhedra, then (π, π_0) is a valid inequality for conv$(\mathcal{P}^1 \cup \mathcal{P}^2)$ if and only if there exist $u^1, u^2 \in \mathbb{R}^m$ such $\pi \leq u^i A^i$ and $\pi_0 \geq u^i b^i$ for $i = 1, 2$.

Valid Inequalities from Disjunctions (Geometric View)



- $\alpha x \geq \beta$ is valid if it is valid for $\mathcal{P} \cap S_1$ and for $\mathcal{P} \cap S_2$.
- Such inequalities are called Split Inequalities (Balas).
- C-G, GMI, MIR, Lift-and-project, k-cuts are special cases of these.
- These can be generated from disjunctions used for branching.
- Conversely, disjunctions used for generating these inequalities can be used for branching.

Lift and Project

- Let's now consider $S = \mathcal{P} \cap \mathbb{B}^n$ and assume that the inequalities $x \leq 1$ are included among those in $Ax \leq b$.
- Note that $conv(S) \subseteq conv(\mathcal{P}_j^0 \cup \mathcal{P}_j^1)$ where $\mathcal{P}_j^0 = \mathcal{P} \cap \{x \in \mathbb{R}^n \mid x_j = 0\}$ and $\mathcal{P}_j^1 = \mathcal{P} \cap \{x \in \mathbb{R}^n \mid x_j = 1\}$ for some $j \in \{1, \ldots, n\}$.
- Applying Proposition 2, we see that the inequality (π, π_0) is valid for $\mathcal{P}_j = conv(\mathcal{P}_j^0 \cup \mathcal{P}_j^1)$ if there exists $u^i \in \mathbb{R}_+^m$, $v^i \in \mathbb{R}_+^n$, and $w^i \in \mathbb{R}_+$ for $i = 0, 1$ such that

$$\begin{aligned} \pi &\leq u^0 A + v^0 + w^0 e_j, \\ \pi &\leq u^1 A + v^1 - w^1 e_j, \\ \pi^0 &\geq u^0 b, \\ \pi^0 &\geq u^1 b - w_1, \end{aligned}$$

- Notice that this is a set of linear constraints, i.e., we could write a linear program to generate constraints based on this disjunction.

The Cut Generating LP

- This leads to the cut generating LP (CGLP), which generates the most violated inequality valid for $\mathcal{P}_j$.

$$\begin{aligned} \min \quad & \pi\hat{x} - \pi^0 \\ \text{subject to} \quad & \pi \leq uA + u^0 e_j, \\ & \pi \leq vA - v^0 e_j, \\ & \pi^0 \geq ub, \\ & \pi^0 \geq vb - v_0, \\ & \sum_{i=1}^{m} u_i + u_0 + \sum_{i=1}^{m} v_i + v_0 = 1 \\ & u, u_0, v, v_0 \geq 0 \end{aligned}$$

- The last constraint is just for normalization.
- This shows that the separation problem for $\mathcal{P}_j$ is polynomially solvable.

Another Derivation using the RLT

So why is it called lift and project?

- Consider the following procedure:
  1. Select $j \in \{1, \ldots, n\}$.
  2. Generate the nonlinear system $x_j(Ax - b) \geq 0$, $(1 - x_j)(Ax - b) \geq 0$.
  3. Linearize the system by substituting y_i for $x_i x_j, i \neq j$, and x_j for x_j^2. Call this polyhedron M_j.
  4. Project M_j into the space of the original variables.
- In this case, the resulting polyhedron is again $\mathcal{P}_j$.
- This procedure is also called the *relaxation linearization technique*.
- It can be strengthened in a number of different ways.

The Lift-and-Project Closure

- The lift-and-project closure is

$$\mathcal{P}^1 = \cap_{j=1}^n \mathcal{P}_j$$

- We have just shown that optimization over the lift-and-project closure can be accomplished in polynomial time.
- Let $\mathcal{P}^k$ be the lift-and-project closure of $\mathcal{P}^{k-1}$ for $k > 1$.
- The lift-and-project rank of $\mathcal{P}$ is the smallest number k such that $\mathcal{P}^k = \text{conv}(S)$.
- Surprisingly, the lift-and-project rank is bounded by n.
- Note that these results apply only to binary and mixed binary integer programs.

The Chvátal-Gomory Procedure

- Let $A = (a_1, a_2, \ldots, a_n)$ and $N = \{1, \ldots, n\}$.
  1. Choose a weight vector u.
  2. Obtain the valid inequality $\sum_{j \in N}(ua_j)x \leq ub$.
  3. Round the coefficients down to obtain $\sum_{j \in N}(\lfloor ua_j \rfloor)x \leq ub$.
  4. Finally, round the right hand side down to obtain the valid inequality

$$\sum_{j \in N}(\lfloor ua_j \rfloor)x \leq \lfloor ub \rfloor$$

- This procedure is called the *Chvátal-Gomory* rounding procedure, or simply the *C-G procedure*.
- Surprisingly, any inequality valid for $\text{conv}(S)$ can be produced by a finite number of iterations of this procedure!

Gomory Cuts From Disjunction

- Consider again the set of solutions to an IP with one equation.
- This time, we write S equivalently as

$$S = \left\{ x \in \mathbb{Z}^n_+ \mid \sum_{j:f_j \le f_0} f_j x_j + \sum_{j:f_j > f_0} (f_j - 1) x_j = f_0 + k \text{ for some integer k} \right\}$$

- Since $k \le -1$ or $k \ge 0$, we have the disjunction

$$\sum_{j:f_j \le f_0} \frac{f_j}{f_0} x_j - \sum_{j:f_j > f_0} \frac{(1 - f_j)}{f_0} x_j \ge 1$$

OR

$$-\sum_{j:f_j \le f_0} \frac{f_j}{(1 - f_0)} x_j + \sum_{j:f_j > f_0} \frac{(1 - f_j)}{(1 - f_0)} x_j \ge 1$$

The Gomory Mixed Integer Cut

- Applying Proposition 1, we get

$$\sum_{j:f_j \le f_0} \frac{f_j}{f_0} x_j + \sum_{j:f_j > f_0} \frac{(1-f_j)}{(1-f_0)} x_j \ge 1$$

- This is called a *Gomory mixed integer* (GMI) cut.
- GMI cuts dominate the associated Gomory cut in general and can also be obtained easily from the tableau.
- In the case of the mixed integer set

$$S = \left\{ x \in \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p} \mid \sum_{j=1}^{p} a_j x_j + \sum_{j=p+1}^{n} g_j x_j = a_0 \right\},$$

the GMI cut is

$$\sum_{j:f_j \le f_0} \frac{f_j}{f_0} x_j + \sum_{j:f_j > f_0} \frac{(1-f_j)}{(1-f_0)} x_j + \sum_{j:g_j > 0} \frac{g_j}{f_0} x_j - \sum_{j:g_j < 0} \frac{g_j}{(1-f_0)} x_j \ge 1$$

The GMI closure

- A GMI cut with respect to a polyhedron $\mathcal{P}$ is any cut that can be derived using the above procedure starting from any inequality valid for $\mathcal{P}$.
- The GMI closure is obtained by adding all GMI cuts to the description of $\mathcal{P}$.
- The GMI closure is a polyhedron, but optimizing over it is an $\mathcal{NP}$-hard problem in general.
- It follows that determining whether there is a GMI cut violated by an arbitrary vector is an $\mathcal{NP}$-complete problem.
- Nevertheless, we have just shown that separation of vectors that are basic feasible solutions to a given LP relaxation from the GMI closure can be accomplished in polynomial time.
- The *GMI rank* of both valid inequalities and polyhedra can be defined in a fashion similar to that of the C-G rank (more on this later).

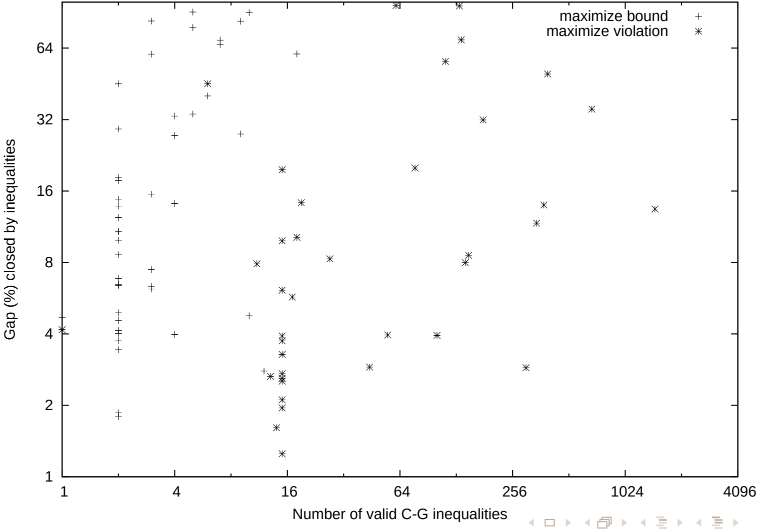
Gomory Cuts vs. Lift-and-Project Cuts

- Note that all Gomory cuts are lift-and-project cuts.
- In fact, there is a direct correspondence between basic feasible solutions of the CGLP and basic (possibly infeasible) solutions of the usual LP relaxation.
- By pivoting in the LP relaxation, we can implicitly solve the cut generating LP (see Balas and Perregaard).
- Thus, the procedure for generating lift-and-project cuts is almost exactly the same as that for generating Gomory cuts.

Selecting Disjunctions for Cutting

- We typically select valid inequalities based on desgree of violation, e.g., the CGLP.
- This is in contrast with the way we select disjunctions for branching.
- What happens if we select valid inequalities based on bound improvement instead of degree of violation?

Comparing Methods for Selecting C-G inequalities



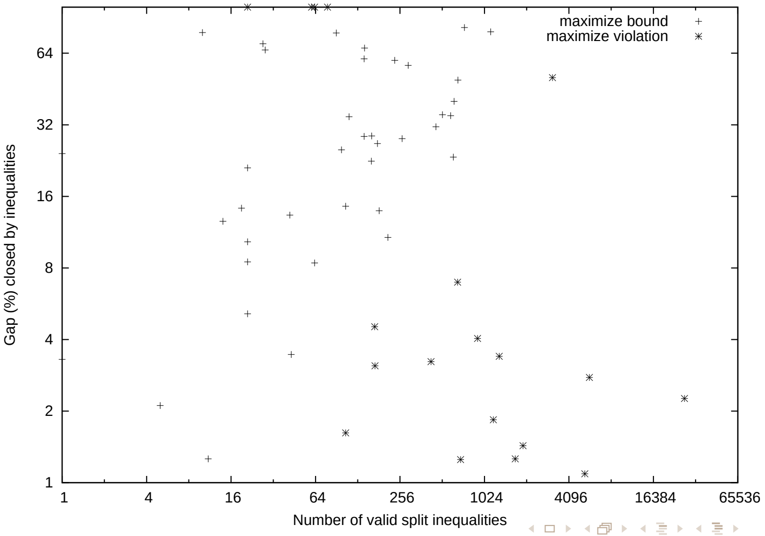
Split Inequalities

- Let $\pi \in \mathbb{Z}^n_+$ and $\pi_0 \in \mathbb{Z}$ be given and define

$$\mathcal{P}^1 = \mathcal{P} \cap \{x \in \mathbb{R}^n \mid \pi x \leq \pi_0\}$$
$$\mathcal{P}^2 = \mathcal{P} \cap \{x \in \mathbb{R}^n \mid \pi x \geq \pi_0 + 1\}$$

- Any inequality valid for $\text{conv}(\mathcal{P}_1 \cup \mathcal{P}_2)$ is valid for S and is called a *split cut*.
- The *split closure* is the set of points satisfying all possible split cuts and is a polyhedron.
- In fact, the split closure and the GMI closure discussed earlier are identical.
- We can define the *split rank* of an inequality and of a polyhedron as before.
- In the pure integer case, the split rank (and GMI rank) of $\mathcal{P}$ is finite, but it may not be in the mixed case.
- In the mixed binary case, the split rank is bounded by n.

Comparing Methods for Selecting Split Inequalities



Towards a Unified Framework

- All of this raises the question of whether disjunction selection can be treated in a unified way for both branching and cutting.
  - Select a violated disjunction according to some criteria.
  - Decide whether to branch or cut.
- The realtionship between branching and cutting with the same disjunction is not well understood.
- How to decide whether to branch or cut is a question largely unaddressed in the literature.
- These is a fascinating and wide open field of research.

Concluding remarks

- Branching and/or cutting using general disjunctions can have a framatic impact on the size of the branch-and-bound tree.
- It is still a very difficult problem to solve.
- We have made some in-roads to practical methods by considering restricted classes of disjunctions.
- There are still many open questions to be answered.

Available at optimization-online:

- A.M. and T.K. Ralphs, *Experiments with Branching using General Disjunctions*, Proceedings of the Eleventh INFORMS Computing Society Meeting, 2009.
- A.M. and T.K. Ralphs, *On the complexity of selecting branching disjunctions for integer programming*, to appear in SIAM Journal of Optimization.