Algorithms for Mixed Integer Bilevel Programs

Exploiting the Lower-level Value Function

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Linear Programming

A linear program (LP) is the problem of minimizing a linear objective over the feasible region

$$\mathcal{S}_{LP} = \left\{x \in \mathbb{R}^n_+ \mid Ax \geq b\right\},$$

defined by $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{R}^m$. That is, the goal of linear programming is to determine

Linear Program

$$\bar{z}_{LP} = \min_{x\in\mathcal{S}_{LP}} cx, \tag{LP}$$

where $c \in \mathbb{Q}^n$.

- Changing any member of the triple (A, b, c) yields a perturbation of (LP).
- We focus on changes to the right-hand-side vector b.

Parametric LP

Considering all possible right-hand-sides yields the *parameterized* version of (LP):

Parametric Linear Program

$$z_{LP}(v) = \min_{x \in \mathcal{S}_{LP}(v)} cx, \qquad \text{(LP}(v)\text{)}$$

where

$$\mathcal{S}_{LP}(v) = \left\{x \in \mathbb{R}^n_+ \mid Ax \geq v\right\},$$

for all $v \in \mathbb{R}^n$.

$z_{LP} : \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$ is the LP *value function*.

- For each $d \in \mathbb{R}^m$, z_{LP} returns the optimal value of (LP).
- z_{LP} is piecewise linear and convex over $\Omega_{LP} = \{v \in \mathbb{R}^m \mid \mathcal{S}_{LP}(v) \neq \emptyset\}$.

Mixed Integer Linear Programming

A mixed integer linear program (MILP) is a well-known generalization of LP.

- In MILP, a specified subset of the variables is required to take on integer values.
- Let this subset be indexed 1 through $p \leq n$.

Formally, the goal of mixed integer linear programming is to determine

Mixed Integer Linear Program

$$z_{IP} = \min_{x \in \mathcal{S}_{IP}} cx, \qquad \text{(MILP)}$$

where

$$\mathcal{S}_{IP} = \left\{ x \in \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p} \mid Ax \geq b \right\},$$

and (A, b, c) are defined as before.

Parametric MILP

The parameterized version of (MILP) is defined as:

Parametric Mixed Integer Linear Program

$$z_{IP}(v) = \min_{x \in \mathcal{S}_{IP}(v)} cx, \qquad \text{(MILP}(v))$$

where

$$\mathcal{S}_{IP}(v) = \left\{x \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+ \mid Ax \geq v\right\},$$

for all $v \in \mathbb{R}^n$.

As in the LP case, we refer to $z_{IP} : \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$ as the *value function*.

- By convention, we say $z_{IP} = \infty$ if $d \notin \Omega_{IP} = \{v \in \mathbb{R}^m \mid \mathcal{S}_{IP}(v) \neq \emptyset\}$ and $z_{IP} = -\infty$ if the objective value is unbounded.
- z_{IP} is piecewise polyhedral but *nonconvex*.

Multilevel Programming

- Multilevel programming is a generalization of traditional mathematical programming that applies to *hierarchical decision systems*.
- In a *multilevel program*:
  - The variables are divided into groups controlled by separate DMs.
  - The constraints of each DM involve the variables of DMs at higher levels in the hierarchy.
  - The DMs have independent, possibly conflicting objectives.
- Conceptually, the variables are fixed sequentially in accordance with the inherent system hierarchy.
- We assume *individual rationality* of the DMs.
  - DMs will be able to predict the reaction of other DMs to decisions made above them.
  - We can collapse the hierarchy into a single optimization model in which the decisions made at the highest level determine the system outcome.

Bilevel Programming

We focus on *bilevel programs*, described as follows.

- $x \in X \subseteq \mathbb{R}^{n_1}_+$ are the *upper-level variables*
- $y \in Y \subseteq \mathbb{R}^{n_2}_+$ are the *lower-level variables*

Bilevel Program

$$\min \left\{ c^1 x + d^1 \bar{y} \mid x \in \mathcal{P}_U \cap X, \bar{y} \in \operatorname{argmin}\{ d^2 y \mid y \in \mathcal{S}_L(x) \cap Y \} \right\} \tag{BP}$$

The *upper-* and *lower-level feasible regions* are

$$\mathcal{P}_U = \left\{ x \in \mathbb{R}^{n_1}_+ \mid A^1 x \geq b^1 \right\}$$

and

$$\mathcal{S}_L(x) = \left\{ y \in \mathbb{R}^{n_2}_+ \mid G^2 y \geq b^2 - A^2 x \right\}.$$

Notation

We utilize the following notation:

Notation

$$
\begin{aligned}
\Omega &= \left\{(x,y) \in \mathbb{R}_+^{n_1} \times \mathbb{R}_+^{n_2} \mid x \in \mathcal{P}_U, y \in \mathcal{S}_L(x)\right\} \\
\Omega^I &= \Omega \cap X \times Y \\
M(x) &= \operatorname{argmin}\{d^2 y \mid y \in \mathcal{S}_L(x)\} \\
M^I(x) &= M(x) \cap Y \\
\mathcal{F} &= \{(x,y) \mid x \in \mathcal{P}_U, y \in M(x)\} \\
\mathcal{F}^I &= \left\{(x,y) \mid x \in \mathcal{P}_U^I, y \in M^I(x)\right\}
\end{aligned}
$$

We can think of a bilevel program as a traditional optimization problem in which a parametric program has been embedded.

Technical Assumptions

We make the following assumptions in order to ensure the problem is well-posed and has a solution.

Assumptions

1. For every action by the leader, the follower has a rational reaction ($\mathcal{S}_L(x) \cap Y \neq \emptyset$ for all $x \in \mathcal{P}_U \cap X$).
2. The follower is semi-cooperative (the leader may choose among alternative members of $M^I(x)$).
3. The feasible set $\mathcal{F}^I$ is nonempty and compact.

The BP can be simply stated as:

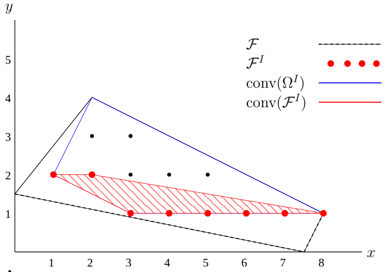
Bilevel Program

$$\max_{(x,y)\in\mathcal{F}^I} c^1 x + d^1 y. \tag{BP}$$

Bilevel Programming Feasible Regions

The following instance of (BP) is from Moore and Bard (1990).

$$
\begin{aligned}
\min_{x \in X} \quad & -x - 10\bar{y} \\
\text{subject to} \quad & \bar{y} \in \operatorname{argmin} \{ y : -25x + 20y \leq 30 \\
& \qquad x + 2y \leq 10 \\
& \qquad 2x - y \leq 15 \\
& \qquad 2x + 10y \geq 15 \\
& \qquad y \in Y \ \}
\end{aligned}
$$



From the figure, we can make several observations:

1. $\mathcal{F} \subseteq \Omega$, $\mathcal{F}^I \subseteq \Omega^I$, and $\Omega^I \in \Omega$
2. $F^I \not\subseteq \mathcal{F}$
3. In this instance, $z_{IP}(v), v \in \{1, \ldots, 8\}$ is identical to F^I

This motivates our study of the value function of the lower-level program.

Contrast with Mixed Integer Linear Programming

In algorithms for solving MILPs, we frequently use the following properties.

Properties

1. If the continuous relaxation has no feasible solution, then neither does the original problem.
2. If the continuous relaxation has a solution, then its objective value is a valid lower bound on that of the original problem.
3. If the solution to the continuous relaxation is integral, then it is optimal for the original problem.

Properties 2 and 3 result from the fact that the set of feasible solutions for the original MILP is contained in the feasible set of the relaxation.

THIS IS NOT THE CASE FOR MIBLP

Properties of MIBLPs

In this example:

- Optimizing over $\mathcal{F}$ yields the *integer* solution $(8, 1)$, with the upper-level objective value 18.
- Imposing integrality yields the solution $(2, 2)$, with upper-level objective value 22

From this we can make two important observations:

1. The objective value obtained by relaxing integrality is not a valid bound on the solution value of the original problem since we may have

$$\min_{(x,y)\in\mathcal{F}} c^1x + d^1y > \min_{(x,y)\in\mathcal{F}^I} c^1x + d^1y.$$

2. Even when solutions to $\min_{(x,y)\in\mathcal{F}} c^1x + d^1y$ are in $\mathcal{F}^I$, they are not necessarily optimal.

Thus, only *Property 1* remains valid.

Bounding Method

Relaxing integrality conditions *and* the requirement $y \in M^I(x)$ yields the relaxation

$$\min_{(x,y)\in\Omega} c^1x + d^1y. \tag{LR}$$

- The resulting bound can be used in combination with a standard variable branching scheme to yield an algorithm that solves (BP).
- The bound is too weak to be effective on interesting problems.
- We can strengthen the linear relaxation with inequalities valid for $\mathcal{F}^I$ to improve the bound.

Bilevel Feasibility Conditions

1. $(x,y) \in \Omega$,
2. $(x,y) \in X \times Y$, and
3. $y \in M^I(x)$.

Pure Integer Problems

In the case where $X = \mathbb{Z}^{n_1}$ and $Y =^{n_2}$, we can derive simple valid inequalities to strengthen the relaxation. Let

$$A := \begin{bmatrix} A^1 \\ A^2 \end{bmatrix}, \quad G := \begin{bmatrix} 0 \\ G^2 \end{bmatrix}, \quad \text{and} \quad b := \begin{bmatrix} b^1 \\ b^2 \end{bmatrix}.$$

A basic feasible solution $(\hat{x}, \hat{y}) \in \Omega^I$ to (LR) is the *unique* solution to

$$a_i' x + g_i' y = b_i, \quad i \in I$$

where I is the set of active constraints at $(\hat{x}, \hat{y})$.

This implies that

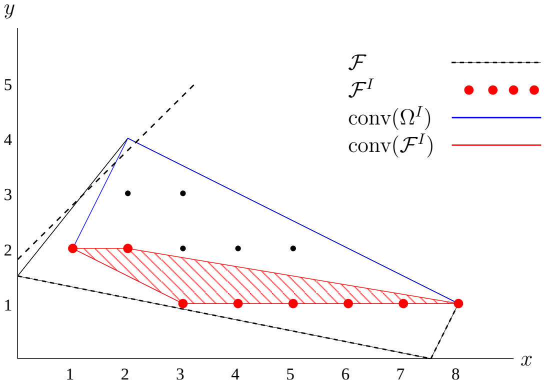
$$\left\{ (x,y) \in \Omega^I \mid \sum_{i \in I} a_i' x + g_i' y = \sum_{i \in I} b_i \right\} = \left\{ (\hat{x}, \hat{y}) \right\}$$

and $\sum_{i \in I} a_i' x + g_i' y \leq \sum_{i \in I} b_i$ is valid for Ω.

Pure Integer Problems (cont.)

A Valid Inequality

$$\sum_{i \in I} a_i' x + g_i' y \leq \sum_{i \in I} b_i - 1 \text{ is valid for } \Omega^I \setminus \{(x, y)\}.$$



We can see from the figure that these cuts are fairly weak.

Value Function Reformulation

Let

$$z_L(x) = \min_{y \in \mathcal{S}_L(x) \cap Y} d^2 y.$$

We can rewrite (BP) as:

$$\begin{aligned} \min \quad & c^1 x + d^1 y \\ \text{subject to} \quad & A^1 x \geq b^1 \\ & A^2 x + G^2 y \geq b^2 \\ & d^2 y = z_L(x) \\ & x \in X, y \in Y. \end{aligned}$$

1. Determining the structure of z_L is very difficult in general.
2. We can derive approximations of the value function

Upper Approximations

We saw, in the previous talk, how to construct upper-approximations.

- We adopt all assumptions given there in what follows here.
- Let y^* be an optimal solution to

$$\min_{y \in \mathcal{S}_L} d^2 y$$

with right-hand-side $x = \bar{x}$.

- One upper-bounding function is given by:

$$f(x) = d_I^2 y_I^* + z_{L_C}(b^2 - A^2 x - G_I^2 y_I^*),$$

where $I = \{1, \ldots p_2\}$ and $C = \{p_2 + 1, \ldots, n_2\}$ and z_{L_C} is the value function of the continuous relaxation of the lower-level problem.

- This bound is *strong*.

We can apply this upper bound in several ways.

Applying the Approximation

One way to apply the bounding function is generate a series of such upper bounds at different right-hand-sides.

- Replacing $d^2y = z_L(x)$ with

$$d^2y \leq f(x),$$

  results in a relaxation that would produce a bound.

- Let $\mathcal{H}^U$ be the set of upper-bounding functions for each $x \in U \subset \mathbb{R}^{m_2}$, for some collection of upper-level solutions.

Then, we have the following relaxation of (BP):

$$\begin{aligned} \min \quad & c^1x + d^1y \\ \text{subject to} \quad & A^1x \geq b^1 \\ & A^2x + G^2y \geq b^2 \\ & d^2y \leq \min_{f \in \mathcal{H}^U} f(x) \\ & x \in X, y \in Y. \end{aligned}$$

Generating "enough" of these functions yields the original problem.

Applying the Approximation (cont.)

Alternatively, we can use this bounding function to generate valid disjunctions to be used in a branch-and-bound or branch-and-cut algorithm.

- Note $z_{L_C} = \max_{u \in V}\{u(b^2 - A^2x)\}$, where V is the set of extreme points of the dual polytope.
- Thus, for each x, f corresponds to some dual feasible basis u.

This allows us to create the disjunction of the form:

Disjunction

$$\left\{ \begin{array}{rcl} u & = & u_1 \\ d^2y & \leq & f_1(x) \end{array} \right\} \text{ OR } \cdots \text{ OR } \left\{ \begin{array}{rcl} u & = & u_{|V|} \\ d^2y & \leq & f_{|V|}(x) \end{array} \right\}.$$

We illustrate this method in for the case of a lower-level problem with a single constraint next.

Lower-level Problems with a Single Constraint

Suppose the lower-level problem contains only a single constraint.

$$\mathcal{S}_L(x) = \left\{ y \in \mathbb{R}_+^{n_2} \mid g^2 y = b^2 - a^2 x \right\}.$$

Let $C^+ = \{i \in C \mid g_i^2 > 0\}$ and $C^- = \{i \in C \mid g_i^2 < 0\}$, and

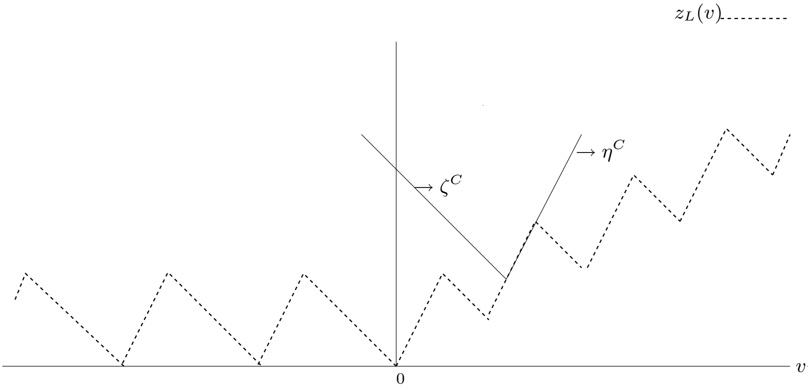
$$\eta^C = \min \left\{ \frac{d_i^2}{g_i^2} \mid i \in C^+ \right\} \text{ and } \zeta^C = \max \left\{ \frac{d_i^2}{g_i^2} \mid i \in C^- \right\}.$$

Then, this disjunction reduces to:

Bilevel Feasibility Disjunction

$$\begin{array}{rcl} a^2 x & \geq & a^2 \bar{x} \\ \zeta^C a^2 x + d^2 y & \leq & \zeta^C a^2 \bar{x} + d^2 y^* \end{array} \quad \text{OR} \quad \begin{array}{rcl} a^2 x & \leq & a^2 \bar{x} \\ \eta^C a^2 x + d^2 y & \leq & \eta^C a^2 \bar{x} + d^2 y^* \end{array}$$

Illustrating the disjunction



This can immediately be used to develop a stronger branching scheme when solutions $(\bar{x}, \bar{y}) \in \Omega^I$ such that $\bar{y} \notin M^I(\bar{x})$ are found.

A Disjunctive Cut Approach

Consider the two polyhedra that result if we impose this disjunction on the original set of constraints in Ω. This yields the polyhedra:

$$P^1 = \left\{ \begin{array}{lcl} A^1 x & \geq & b^1 \\ a^2 x + g^2 y & = & b^2 \\ a^2 x & \geq & a^2 \bar{x} \\ -\zeta^C a^2 x - d^2 y & \geq & -\zeta^C a^2 \bar{x} - d^2 y^* \\ x, y & \geq & 0 \end{array} \right\}$$

and

$$P^2 = \left\{ \begin{array}{lcl} A^1 x & \geq & b^1 \\ a^2 x + g^2 y & = & b^2 \\ -a^2 x & \geq & -a^2 \bar{x} \\ -\eta^C a^2 x - d^2 y & \geq & -\eta^C a^2 \bar{x} - d^2 y^* \\ x, y & \geq & 0. \end{array} \right\}$$

Constructing the Disjunctive Cut

Let (u^i, v^i, w^i, z^i) be multipliers for the constraints in polyhedron P^i. The following inequalities are valid for P^1:

$$u^1A^1x + v^1a^2x + w^1a^2x - z^1\zeta^C a^2x + v^1g^2y - z^1d^2y \geq$$
$$u^1b^1 + v^1b^2 + w^1a^2\bar{x} - z^1(\zeta^C a^2\bar{x} + d^2y^*)$$

and P_2:

$$u^2A^1x + v^2a^2x - w^2a^2x - z^2\eta^C a^2x + v^2g^2y - z^2d^2y \geq$$
$$u^2b^1 + v^2b^2 - w^2a^2\bar{x} - z^2(\eta^C a^2\bar{x} + d^2y^*).$$

It is well-known that, given these inequalities, we can construct an inequality $\alpha x + \beta y \geq \gamma$ that is valid for $\text{conv}(P^1 \cup P^2)$ by selecting α, β, and γ such that

$$\alpha \geq \max\{\pi_1^1, \pi_1^2\}, \quad \beta \geq \max\{\pi_2^1, \pi_2^2\}, \quad \text{and} \quad \gamma \leq \min\{\pi_0^1, \pi_0^2\}.$$

Linear Description of the Set of Valid Inequalities

Thus, the inequality $\alpha x + \beta y \geq \gamma$ is valid for $\text{conv}(P^1 \cup P^2)$ if

$$\begin{aligned}
\alpha - (u^{1^+} - u^{1^-})A^1 - (v^{1^+} - v^{1^-})a^2 - w^1 a^2 + z^1 \zeta^C a^2 &\geq 0 \\
\alpha - (u^{2^+} - u^{2^-})A^1 - (v^{2^+} - v^{2^-})a^2 + w^2 a^2 + z^2 \eta^C a^2 &\geq 0 \\
\beta - (v^{1^+} - v^{1^-})g^2 + z^1 d^2 &\geq 0 \\
\beta - (v^{2^+} - v^{2^-})g^2 + z^2 d^2 &\geq 0 \\
\gamma - (u^{1^+} - u^{1^-})b^1 - (v^{1^+} - v^{1^-})b^2 - w^1 a^2 \bar{x} + z^1(\zeta^C a^2 \bar{x} - d^2 y^*) &\leq 0 \\
\gamma - (u^{2^+} - u^{2^-})b^1 - (v^{2^+} - v^{2^-})b^2 + w^2 a^2 \bar{x} + z^2(\eta^C a^2 \bar{x} - d^2 y^*) &\leq 0 \\
u^{1^+}, u^{1^-}, u^{2^+}, u^{2^-}, v^{1^+}, v^{2^-}, w^1, w^2, z^1, z^2 &\geq 0.
\end{aligned}$$

Cut Generating LP

To find the deepest cut, we can solve the *cut generation LP*:

$$
\begin{array}{ll}
\min & \alpha\bar{x} + \beta\bar{y} - \gamma \\
\text{s.t.} & \alpha - (u^{1^+} - u^{1^-})A^1 - (v^{1^+} - v^{1^-})A^2 - w^1 a^2 + z^1 \zeta^C a^2 \geq 0 \\
& \alpha - (u^{2^+} - u^{2^-})A^1 - (v^{2^+} - v^{2^-})a^2 + w^2 A^2 + z^2 \eta^C a^2 \geq 0 \\
& \beta - (v^{1^+} - v^{1^-})g^2 + z^1 d^2 \geq 0 \\
& \beta - (v^{2^+} - v^{2^-})g^2 + z^2 d^2 \geq 0 \\
& \gamma - (u^{1^+} - u^{1^-})b^1 - (v^{1^+} - v^{1^-})b^2 - w^1 a^2 \bar{x} + z^1(\zeta^C a^2 \bar{x} + d^2 y^*) \leq 0 \\
& \gamma - (u^{2^+} - u^{2^-})b^1 - (v^{2^+} - v^{2^-})b^2 + w^2 a^2 \bar{x} + z^2(\eta^C a^2 \bar{x} + d^2 y^*) \leq 0 \\
& u^{1^+} + u^{1^-} + v^{1^+} + v^{1^-} + w^1 + z^1 + u^{2^+} + u^{2^-} + v^{2^+} + v^{2^-} + w^2 + z^2 = 1 \\
& u^{1^+}, u^{1^-}, u^{2^+}, u^{2^-}, v^{1^+}, v^{1^-}, v^{2^+}, v^{2^-}, w^1, w^2, z^1, z^2 \geq 0,
\end{array}
$$

similar to that used in constructing lift-and-project cuts.

Current Work

We are currently working on developing methods:

- Reduce the size of the disjunction in the general case
- Employ the lower approximations in our algorithmic framework
- Prepocess the problems to increase the algorithm's speed.

References I

Moore, J. and J. Bard 1990. The mixed integer linear bilevel programming problem. *Operations Research* **38**(5), 911–921.