The Value Function of a Mixed-Integer Linear Program with a Single Constraint

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Abstract

The *value function* of a mixed-integer linear program (MILP) is a function that returns the optimal solution value as a function of the right-hand side. In this paper, we analyze the structure of the value function of a MILP with a single constraint. We show that the value function is uniquely determined by a finite number of break points and at most two slopes. We derive conditions for the value function to be continuous and analyze its behavior where it is discontinuous. We also propose a method for systematically extending the value function from a neighborhood of the origin to the entire real line using the technique of *maximal subadditive extension*.

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1 Introduction

This paper concerns the value function of a mixed-integer linear program (MILP) with a single constraint. The motivation for this study is to gain insight into the structure of the value function of a general MILP by considering the value functions of various single-row relaxations in much the same way as cutting planes can be generated by considering the feasible regions of single-row relaxations. We hope this line of work will eventually lead to the development of methodology for approximating the value functions of general MILPs. Herein, we detail the results of our initial theoretical examination of the structure of the value function of a MILP with a single equality constraint and non-negativity constraints on all variables. We show that the value function is uniquely determined by a finite number of break points and at most two slopes, derive conditions for the value function to be continuous and suggest a method for systematically extending the value function from a specified neighborhood of the origin to the entire real line. Although we focus here specifically on the case of a MILP with a single constraint, many of these results hold in more general settings.

Previous work on the structure of the value function of a MILP has been primarily theoretical and there has been no work to date on the value function of a MILP with a single constraint. In a series of papers, Johnson [1974, 1973, 1979] and later Jeroslow [1978] developed the theory of *subadditive duality* for integer linear programs, thereby extending many familiar concepts from the realm of linear programs (LP) to that of integer programs. After this breakthrough, Blair and Jeroslow [1982] showed that the value function of a pure-integer linear program (PILP) is a *Gomory function* that can be derived by taking the maximum of finitely many subadditive functions. Lasserre [2004a,b] introduced a method of constructing the value function of a PILP using two-sided $\mathbb{Z}$ transformation and Cauchy residue techniques on the generating function representation of the corresponding PILP. Loera et al. [2004a,b] suggested a method of applying Barvinok [1994]'s algorithm for counting lattice points in a polyhedron of fixed dimension to a specially constructed polyhedron that includes, for any right-hand side, the corresponding minimal test set (*reduced Gröbner basis*). For the mixed-integer case, most significant results are contained in the work of Blair [1995], who showed that the value function of a MILP can be written as a *Jeroslow formula*, consisting of a Gomory function and a correction term. See Güzelsoy and Ralphs [2007] for a more detailed review of all previous work.

2 Duality and the Value Function

In what follows, we consider a MILP with one equality constraint, which is the problem of determining

$$z_P = \min_{x \in \mathcal{S}} cx, \tag{P}$$

where $c \in \mathbb{R}^n$ is the objective function vector and $\mathcal{S} = \{x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+ \mid ax = b\}$ is the feasible region, defined by $a \in \mathbb{Q}^n$, $b \in \mathbb{R}$, as well as a scalar r indicating the number of integer variables. A vector $x \in \mathcal{S}$ is called a *feasible solution* and has the *solution value* cx. Any $x^* \in \mathcal{S}$ such that $cx^* = z_P$ is called an *optimal* *solution*. We refer to the MILP (P) as the *primal problem*.

We are concerned with the *value function* of (P). Formally, the value function is a function $z : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ that returns the optimal solution value of a MILP as a function of the right-hand side. In our case, we have

$$z(d) = \min_{x \in \mathcal{S}(d)} cx, \tag{1}$$

where for a given $d \in \mathbb{R}$, $\mathcal{S}(d) = \{x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+ \mid ax = d\}$. For the rest of the paper, we let $I = \{1, \ldots, r\}$, $C = \{r+1, \ldots, n\}$, $N = I \cup C$, and for any index set $D \subseteq N$, and a vector y indexed on N, we denote by y_D the sub-vector consisting of the corresponding components of y.

By convention, we let $z(d) = \infty$ if $\mathcal{S}(d) = \emptyset$. To simplify the presentation, we make a number of assumptions. First, we assume that $z(0) = 0$, since otherwise $z(d) = -\infty$ for all $d \in \mathbb{R}$ with $\mathcal{S}(d) \neq \emptyset$. Although our results remain valid when either $a \in \mathbb{Q}^n_+$ or $a \in \mathbb{Q}^n_-$, we assume that both $N^+ = \{i \in N \mid a_i > 0\}$ and $N^- = \{i \in N \mid a_i < 0\}$ are nonempty since otherwise, $z(d) = \infty \; \forall d \in \mathbb{R}_+ \backslash \{0\}$ in the first case and $\forall d \in \mathbb{R}_- \backslash \{0\}$ in the latter case. Finally, we assume $r < n$, that is, at least one of $C^+ = \{i \in C \mid a_i > 0\}$ and $C^- = \{i \in C \mid a_i < 0\}$ is nonempty, to ensure that $S(d) \neq \emptyset$ for all $d \in \mathbb{R}$. Taking all of these assumptions together, we have that $-\infty < z(d) < \infty \; \forall d \in \mathbb{R}$. Note that all of these assumptions can be easily relaxed. In particular, the last assumption, which may appear restrictive, is made without loss of generality, since the value function a PILP with a single constraint can always be extended to that of a corresponding MILP with a single constraint in such a way that the two functions agree for all right-hand sides $d \in \mathbb{R}$ for which the original PILP is feasible (Blair and Jeroslow [1977]).

Because of the relevance to our work, we briefly introduce the subadditive dual problem and its relation to the primal problem (P). For a fixed right-hand side $b \in \mathbb{R}$, the *subadditive dual* of the MILP (P) is the problem of determining

$$\begin{aligned} \max \quad & F(b) \\ & F(a_j) \leq c_j \quad j \in I, \\ & \bar{F}(a_j) \leq c_j \quad j \in C, \\ & F(0) = 0 \text{ and } F \text{ is subadditive}^1, \end{aligned} \tag{D}$$

where the function $\bar{F}$, first used by Gomory and Johnson [1972] for cutting plane algorithms, is the *upper d-directional derivative* of F at zero and is defined by

$$\bar{F}(d) = \limsup_{\delta \to 0^+} \frac{F(\delta d)}{\delta} \; \forall d \in \mathbb{R}. \tag{2}$$

We call any function F satisfying the constraints of (D) *dual feasible*. Note that $\bar{F}$ is positively homogeneous, subadditive, and bounds F from above.

As in the LP case, the relationship between the primal problem (P) and dual problem (D) can be summarized by the following theorems. Extensions of other results from linear programming, such as complementarity and optimality conditions, are also possible (see Wolsey [1981]).

[1] A function F is *subadditive* over a domain Θ if $F(\lambda_1) + F(\lambda_2) \geq F(\lambda_1 + \lambda_2)$ for all $\lambda_1, \lambda_2, \lambda_1 + \lambda_2 \in \Theta$

Theorem 1 *(Weak Duality by Jeroslow [1978, 1979], Wolsey [1981]) Let x be a feasible solution to the MILP (P) and let F be a feasible solution to the subadditive dual (D). Then $F(b) \leq cx$.*

Theorem 2 *(Strong duality by Jeroslow [1978, 1979], Wolsey [1981]) If the primal problem (P) (resp., the dual) has a finite optimum, then so does the dual problem (D) (resp., the primal) and they are equal.*

3 Linear Approximations

Unlike in the case of linear programming, computing an optimal solution to (D) is an intractable problem in general. Since linear functions are subadditive, however, one possibility for approximating the value function is to consider only linear functions in (D). Replacing subadditivity with linearity in (D) reduces the dual problem to that of the LP relaxation of the MILP (P), that is, the LP obtained by removing the integrality requirements for the variables indexed by the set I. It follows that the value function of the LP relaxation of the MILP (P) is dual feasible and bounds z from below. Furthermore, it is well-known that in the case of a MILP with a single constraint, the value function of the LP relaxation, which we shall denote by F_L, has a convenient closed form. Let

$$\eta = \min\{\frac{c_i}{a_i} \mid i \in N^+\} \text{ and } \zeta = \max\{\frac{c_i}{a_i} \mid i \in N^-\}. \tag{3}$$

Then we have

$$F_L(d) = \max\{ud \mid \zeta \leq u \leq \eta, u \in \mathbb{R}\} = \begin{cases} \eta d & \text{if } d > 0 \\ 0 & \text{if } d = 0 \\ \zeta d & \text{if } d < 0 \end{cases}. \tag{4}$$

Note that we must have $\eta \geq \zeta$, since otherwise the dual problem of the LP relaxation of (P) is infeasible, which in turn, is a contradiction to our initial assumptions that $a \in \mathbb{Q}^n$ and $z(0) = 0$. Similarly, we can get an upper bound on the value function by considering only the continuous variables. Let

$$\eta^C = \min\{\frac{c_i}{a_i} \mid i \in C^+\} \text{ and } \zeta^C = \max\{\frac{c_i}{a_i} \mid i \in C^-\}. \tag{5}$$

By convention, if $C^+ = \emptyset$, then we set $\eta^C = \infty$. Similarly, if $C^- = \emptyset$, then we set $\zeta^C = -\infty$. The function F_U, defined by

$$F_U(d) = \min\{c_C x_C \mid a_C x_C = d, x_i \geq 0 \;\; \forall i \in C\} = \begin{cases} \eta^C d & \text{if } d > 0 \\ 0 & \text{if } d = 0 \\ \zeta^C d & \text{if } d < 0 \end{cases}, \tag{6}$$

is then an upper bound on z. This follows from the fact that for a given feasible right-hand side, any optimal solution to the LP (6) can be extended to a feasible solution for the original MILP (P) by fixing all the integer variables to 0. In addition, $z(d) = F_U(d) = F_L(d) \; \forall d \in \mathbb{R}_+$ if and only if $\eta = \eta^C$ and similarly, $z(d) = F_U(d) = F_L(d) \; \forall d \in \mathbb{R}_-$ if and only if $\zeta = \zeta^C$.

Example 1 Consider the following MILP instance with a fixed right-hand side b

$$\begin{array}{ll} \min & \frac{1}{2}x_1 + \frac{1}{2}x_3 + 2x_4 + x_5 + \frac{3}{4}x_6 \\ \text{s.t} & x_1 - \frac{3}{2}x_2 + \frac{3}{2}x_3 + x_4 - x_5 + \frac{1}{3}x_6 = b \quad \text{and} \\ & x_1, x_2, x_3 \in \mathbb{Z}_+, x_4, x_5, x_6 \in \mathbb{R}_+. \end{array} \tag{7}$$

We have $\eta = \frac{1}{2}$, $\zeta = 0$, $\eta^C = 2$ and $\zeta^C = -1$. Consequently, we have the following lower and upper bounding functions on the value function of (7) (see Figure 1).

$$F_L(d) = \begin{cases} \frac{1}{2}d & \text{if } d \geq 0 \\ 0 & \text{if } d < 0 \end{cases} \quad \text{and} \quad F_U(d) = \begin{cases} 2d & \text{if } d \geq 0 \\ -d & \text{if } d < 0 \end{cases} \tag{8}$$

□

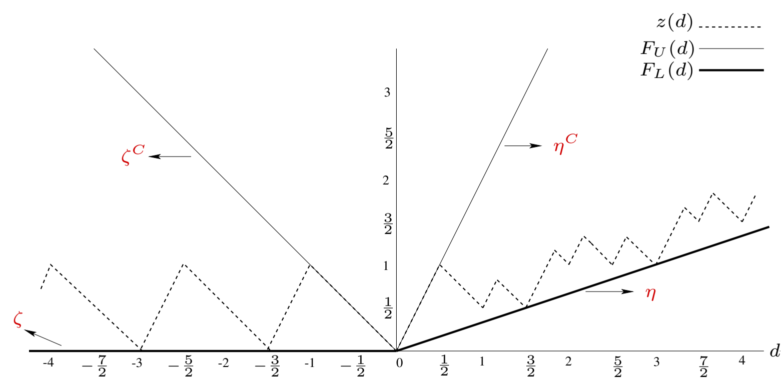

Figure 1: Bounding functions on the value function of (7).

Not only do the functions F_U and F_L bound the value function, but the following proposition shows that the bounds are always tight.

Proposition 3 *Let F_L and F_U be defined as in (4) and (6) and let*

$$\begin{aligned} d_U^+ &= \sup\{d \geq 0 \mid z(d) = F_U(d)\}, \\ d_U^- &= \inf\{d \leq 0 \mid z(d) = F_U(d)\}, \\ d_L^+ &= \inf\{d > 0 \mid z(d) = F_L(d)\}, \ \textit{and} \\ d_L^- &= \sup\{d < 0 \mid z(d) = F_L(d)\}. \end{aligned}$$

Then

(i) $z(d) = F_U(d) \; \forall d \in (d_U^-, d_U^+)$,

(ii) $\eta < \eta^C \iff \{d \mid z(d) = F_U(d) = F_L(d), d \in \mathbb{R}_+\} \equiv \{0\} \iff d_U^+ < \infty$,

(iii) $\zeta > \zeta^C \iff \{d \mid z(d) = F_U(d) = F_L(d), d \in \mathbb{R}_-\} \equiv \{0\} \iff d_U^- > -\infty$,

(iv) $d_L^+ \geq d_U^+$ *if* $d_L^+ > 0$, *and similarly,* $d_L^- \leq d_U^-$ *if* $d_L^- < 0$,

(v) if $b \in \{d \in \mathbb{R} \mid z(d) = F_L(d)\}$, then $z(kb) = kF_L(b) \;\; \forall k \in \mathbb{Z}_+$.

Proof.

(i) We first consider the interval $[0, d_U^+)$. If $\eta^C = \infty$, then $d_U^+ = 0$ because otherwise there would exist $d > 0$ with $z(d) = F_U(d) = \infty$, which is a contradiction to our initial assumption that z is finite everywhere. Therefore, assume that $\eta^C < \infty$ and that for $b_1 > 0$, we have $F_U(b_1) > z(b_1)$. Note that if we can show that $F_U(d) > z(d) \; \forall d > b_1$, then we must have $z(d) = F_U(d) \; \forall d \in [0, d_U^+)$. Let $x^1 \in \mathbb{Z}_+^I \times \mathbb{R}_+^C$ be an optimal solution to (P) with right hand side b_1. Then,

$$F_U(b_1) = \eta^C b_1 > z(b_1) = cx^1. \tag{9}$$

Let $t \in C$ be such that $\eta^C = \frac{c_t}{a_t}$. Then, setting $\gamma = \sum_{\substack{i=1 \\ i \neq t}}^{n} a_i x_i^1$, we have

$$x_t^1 = \frac{b_1 - \gamma}{a_t}.$$

Next, let $b_2 > b_1$. In this case, the vector $x^2 \in \mathbb{Z}_+^I \times \mathbb{R}_+^C$ defined by

$$x_i^2 = x_i^1 \; \forall i \in N, i \neq t \;\text{ and }\; x_t^2 = \frac{b_2 - \gamma}{a_t}$$

is a feasible solution to the MILP with right hand side b_2. This implies that

$$cx^2 = z(b_1) + c_t \left(\frac{b_2 - \gamma}{a_t} - \frac{b_1 - \gamma}{a_t} \right) = z(b_1) + \eta^C (b_2 - b_1) \geq z(b_2).$$

Therefore, by (9), we have

$$F_U(b_2) = \eta^C b_2 = \eta^C b_1 + (\eta^C b_2 - \eta^C b_1) > z(b_1) + \eta^C (b_2 - b_1) \geq z(b_2).$$

Hence, we have the result. Similar arguments can be made for the interval $(d_U^-, 0]$.

(ii) Since F_L and F_U are linear on $\mathbb{R}_+$ and $F_L(d) = \eta d \leq z(d) \leq \eta^C d = F_U(d) \; \forall d \in \mathbb{R}_+$, then it is clear that $\eta < \eta^C \Longleftrightarrow \{d \mid z(d) = F_U(d) = F_L(d), d \in \mathbb{R}_+\} \equiv \{0\}$.

For the remaining part, we will show that $\eta < \eta^C \Longleftrightarrow d_U^+ < \infty$. Now, if $\eta = \eta^C$, then $z(d) = F_U(d) = F_L(d) \; \forall d \in \mathbb{R}_+$ and hence $d_U^+ = \infty$. On the other hand, if $\eta < \eta^C$, then for $t \in I$ such that $\frac{c_t}{a_t} = \eta$, we have

$$c_t \geq z(a_t) \geq \eta a_t = c_t$$

and thus, $F_L(a_t) = z(a_t)$ and $a_t > 0$. If $d_U^+ = \infty$, then from (i), we should also have $z(a_t) = F_U(a_t)$. However, this is a contradiction to our previous assumption that $\eta < \eta^C$.

(iii) The proof follows from arguments similar to those made above.

(iv) Assume that $d_L^+ > 0$ and $d_L^+ < d_U^+$. From (i) and our discussion in the proof of (ii), we then have $z(d) = F_U(d) = F_L(d) \; \forall d \in \mathbb{R}_+$ and hence, $d_L^+ = 0$, which is a contradiction. The proof of the second part follows from similar arguments.

(v) The proof follows from the fact that

$$kF_L(b) = kz(b) \geq z(kb) \geq F_L(kb) = kF_L(b)\ \forall k \in \mathbb{Z}_+.$$

□

It is also worth mentioning that it is not difficult to show that we must have $F_U(d) = \bar{z}(d)\ \forall d \in \mathbb{R}_+$ if $d_U^+ > 0$ and similarly, $F_U(d) = \bar{z}(d)\ \forall d \in \mathbb{R}_-$ if $d_U^- < 0$, where $\bar{z}$ is the upper d-directional derivative of the value function.

Example 2 Consider the MILP instance (7) and its value function given in Figure 1. From Proposition 3, we have $d_U^+ = \frac{1}{2}$, $d_U^- = -1$, $d_L^+ = 1$, $d_L^- = \frac{-3}{2}$,

$$z(d) = \begin{cases} 2d & \forall d \in [0, \frac{1}{2}] \\ -d & \forall d \in [-1, 0] \end{cases},$$

and furthermore, for any $k \in \mathbb{Z}_+$

$$z\left(\frac{3}{2}k\right) = \frac{1}{2}k \text{ and } z\left(\frac{-3k}{2}\right) = 0$$

Observe that these results agree with the value function.

□

4 Structure of the Value Function

Blair and Jeroslow [1977] and Blair [1995] showed that the value function of a general MILP is piecewise-linear and can be written as a *Jeroslow Formula* defined as the value function of a related pure integer program and a linear correction term obtained from the coefficients of the continuous variables. Below, we extend these results for the case under consideration and show that z can be represented with at most two continuous variables and consequently, each linear segment of z can be described by a unique integral vector and a slope obtained from the coefficients of these continuous variables. In addition, we analyze the behavior of z at a right-hand-side d where it is discontinuous and derive the necessary and sufficient conditions for the continuity of z.

As a first step, let $T \subseteq C$ consist of the indices t^+ and t^- of two continuous variables that achieve the ratios η^C and ζ^C, respectively (if they exist). Formally, let $t^+ \in C$ be such that $\eta^C = \frac{c_{t^+}}{a_{t^+}}$ if $\eta^C < \infty$ and similarly, let $t^- \in C$ be such that $\zeta^C = \frac{c_{t^-}}{a_{t^-}}$ if $\zeta^C > -\infty$. Finally, let $T = \{t^+ \mid \eta^C < \infty\} \cup \{t^- \mid \zeta^C > -\infty\}$.

Proposition 4 *Let*

$$\begin{aligned} \nu(d) = \min \quad & c_I x_I + c_T x_T \\ s.t. \quad & a_I x_I + a_T x_T = d \\ & x_I \in \mathbb{Z}_+^I,\ \ x_T \in \mathbb{R}_+^T \end{aligned} \quad . \tag{10}$$

Then, $\nu(d) = z(d)$ for all $d \in \mathbb{R}$.

Proof.

1. $\nu \geq z$: This result follows from the fact that (10) is a restriction of (P).

2. $\nu \leq z$: For a given $b \in \mathbb{R}$, let $\bar{y} \in \mathbb{Z}_+^I$ be the integer part of an optimal solution to MILP (P) with right-hand-side b. Then we have

$$\begin{aligned}
\nu(b) \quad & \leq c_I\bar{y} + \quad \min c_T x \\
& \qquad\qquad a_T x = b - a_I\bar{y} \\
& \qquad\qquad x \in \mathbb{R}_+^T \\
& = c_I\bar{y} + \quad \max\ u(b - a_I\bar{y}) \\
& \qquad\qquad \zeta^C \leq u \leq \eta^C \\
& = c_I\bar{y} + \quad u^*(b - a_I\bar{y}) \\
& = z(b)
\end{aligned}$$

where

$$u^* = \begin{cases} \eta^C & \text{if} \quad b - a_I\bar{y} > 0, \\ \zeta^C & \text{if} \quad b - a_I\bar{y} < 0, \\ 0 & \text{otherwise} \end{cases}$$

The first inequality follows from the fact that $\bar{y}$ has to be the integer part of a feasible solution to (10), because otherwise, the dual problem in the second equation above would be unbounded and hence $\bar{y}$ cannot be an integer part of an optimal solution to MILP (P) with right-hand-side b.

□

Proposition 4 states that we can assume without loss of generality that we have at most two continuous variables, i.e., those indexed by the set T. The rest of the continuous variables can be considered redundant. With this observation, we can simplify the result of Blair [1995] that states that the value function of a general MILP can be written as a *Jeroslow Formula* to the special case under consideration here.

Let $M \in \mathbb{Z}_+$ be such that for any $t \in T$, $\frac{Ma_j}{a_t} \in \mathbb{Z}$ for all $j \in I$. Note that such an integer exists by the rationality of a. Define

$$\begin{aligned}
g(q) = \min \quad & c_I x_I + \tfrac{1}{M} c_T x_T + z(\varphi)v \\
\text{s.t} \quad & a_I x_I + \tfrac{1}{M} a_T x_T + \varphi v = q \\
& x_I \in \mathbb{Z}_+^I, x_T \in Z_+^T, v \in \mathbb{Z}_+
\end{aligned} \tag{11}$$

for all $q \in \mathbb{R}$, where $\varphi = -\frac{1}{M}\sum_{t \in T} a_t$. Furthermore, for $t \in T$, define

$$\omega_t(d) = g(\lfloor d \rfloor_t) + \frac{c_t}{a_t}(d - \lfloor d \rfloor_t) \tag{12}$$

for all $d \in \mathbb{R}$, where $\lfloor d \rfloor_t = \frac{a_t}{M}\left\lfloor \frac{Md}{a_t} \right\rfloor$ and for $\lambda \in \mathbb{R}$, $\lfloor \lambda \rfloor$ is the largest integer with $\lambda \geq \lfloor \lambda \rfloor$.

Theorem 5 *(Blair [1995]) For any $d \in \mathbb{R}$,*

$$z(d) = \min_{t \in T} \omega_t(d). \tag{13}$$

Note that both ω_{t^+} and ω_{t^-} are piecewise linear with finitely many linear segments on any closed interval and each of those linear segments has a slope of η^C and ζ^C, respectively. It then follows that z is also piecewise-linear with finitely many linear segments on any closed interval and furthermore, each of those linear segments either coincides with ω_{t^+} and has a slope of η^C or coincides with ω_{t^-} and has a slope of ζ^C. Below, we formally present results that follow from this observation by analyzing the structure of ω_{t^+} and ω_{t^-}.

Theorem 6

(i) For $t \in T$, let $b \in \mathbb{R}$ be a breakpoint of ω_t. Then $\frac{Mb}{a_t} \in \mathbb{Z}$.

(ii) ω_{t^+} is continuous from the right and ω_{t^-} is continuous from the left.

(iii) ω_{t^+} and ω_{t^-} are both lower-semicontinuous.

Proof.

(i) Let b be given such that $\frac{Mb}{a_t} \notin \mathbb{Z}$ and let

$$U \equiv \begin{cases} \left[\lfloor b \rfloor_t, \lfloor b \rfloor_t + \frac{a_t}{M}\right), & \text{if } t = t^+ \\ \left(\lfloor b \rfloor_t + \frac{a_t}{M}, \lfloor b \rfloor_t\right], & \text{if } t = t^- \end{cases}.$$

Observe that for any $d \in U$, we have $\lfloor d \rfloor_t = \lfloor b \rfloor_t$ and ω_t is linear over U. Since $b \in U$ and $b \neq \lfloor b \rfloor_t$, we must have that ω_t is continuous at b and therefore b cannot be a breakpoint.

(ii) From the proof of (i), for any $d \in \mathbb{R}$, ω_{t^+} is linear over the interval $\left[\lfloor d \rfloor_t, \lfloor d \rfloor_t + \frac{a_t}{M}\right)$ and therefore continuous from the right. Similarly, ω_{t^-} is linear over the interval $\left(\lfloor d \rfloor_t + \frac{a_t}{M}, \lfloor d \rfloor_t\right]$ and therefore continuous from the left.

(iii) Let $b \in \mathbb{R}$ be a breakpoint of ω_{t^+}. From our discussion in the proof of (i),

$$\omega_{t^+}(d) = \begin{cases} g\left(b - \frac{a_{t^+}}{M}\right) + \frac{c_{t^+}}{a_{t^+}}\left(d - \left(b - \frac{a_{t^+}}{M}\right)\right) & d \in \left[b - \frac{a_{t^+}}{M}, b\right), \\ g(b) + \frac{c_{t^+}}{a_{t^+}}(d - b) & d \in \left[b, b + \frac{a_{t^+}}{M}\right). \end{cases}$$

Then we have

$$\begin{aligned} \lim_{\rho \to b^-} \omega_{t^+}(\rho) &= g\left(b - \frac{a_{t^+}}{M}\right) + \frac{c_{t^+}}{a_{t^+}}\left(b - \left(b - \frac{a_{t^+}}{M}\right)\right) \\ &= g\left(b - \frac{a_{t^+}}{M}\right) + \frac{c_{t^+}}{M} \\ &\geq g(b) \\ &= \omega_{t^+}(b), \end{aligned}$$

where the inequality follows the fact that if $(\bar{x}_I, \bar{x}_T, \bar{v})$ is an optimal solution to (11) with right-hand-side $b - \frac{a_{t^+}}{M}$, then $\left(\bar{x}_I, x_{t^+} + 1, \bar{x}_{T \setminus t^+}, \bar{v}\right)$ would be a feasible solution for right-hand-side b and the last equality follows from the fact that $b = \lfloor b \rfloor_{t^+}$ by (i). Since from (ii), ω_{t^+} is continuous from the right, we conclude that $\lim_{\rho \to b} \omega_{t^+}(\rho) \geq \omega_{t^+}(b)$ and therefore, ω_{t^+} is lower-semicontinuous.

One can show that ω_{t^-} is also lower-semicontinuous with similar arguments.

□

Corollary 7

(i) (Meyer [1975]) z is lower-semicontinuous.

(ii) If z is discontinuous at $b \in \mathbb{R}$, then there exists a $\bar{y} \in \mathbb{Z}^I_+$ such that $b - a_I\bar{y} = 0$.

(iii) Let U be a maximal interval on which the value function z is linear. If $z(d) = \omega_{t^+}(d)\ \forall d \in U$, then U is closed from the left. On the other hand, if $z(d) = \omega_{t^-}(d)\ \forall d \in U$, then U is closed from the right.

Proof.

(i) From Theorems 5 and 6, z is the minimum of two piecewise-linear, lower-semicontinuous functions and therefore, z is also lower-semicontinuous.

(ii) Assume without loss of generality that $b - a_I\bar{y} > 0$. Then we have,

$$\begin{aligned} z(b) &= \lim_{\rho \to b}(c_I\bar{y} + \eta^C(\rho - a_I\bar{y})) \\ &\geq \lim_{\rho \to b} z(\rho) \\ &\geq z(b), \end{aligned}$$

where the first inequality follows from the fact that $\bar{y}$ is the integral part of a feasible solution for any $d \in [a_I\bar{y}, \infty)$ and the second inequality from (i) that z is lower-semicontinuous.

However, this is a contradiction to the initial statement that z is discontinuous at b since at least one of $\lim_{\rho \to b^+} z(\rho) > z(b)$ and $\lim_{\rho \to b^-} z(\rho) > z(b)$ has to be satisfied.

(iii) Assume that $z(d) = \omega_{t^+}(d)\ \forall d \in U$, U is not closed from the left and let $b = \inf\{d \mid d \in U\}$. Note that we must have $z(b) < \omega_{t^+}(b)$ due to the fact that ω_{t^+} is continuous from the left by Theorem 6 and therefore is linear over $\{b\} \cup U$. Since z is discontinuous at b, there exists by (ii) a $\bar{y} \in \mathbb{Z}^I_+$ such that $b - a_I\bar{y} = 0$. Then for each $t \in T$, we must have $\frac{Mb}{a_t} \in \mathbb{Z}$, $b = \lfloor b \rfloor_t$, and consequently, $z(b) = \omega_t(b)$, which is a contradiction.

Similar arguments can be made to show that U is closed from the right when $z(d) = \omega_{t^-}(d)\ \forall d \in U$.

□

Theorem 8 *If the value function z is linear on an interval $U \subset \mathbb{R}$, then there exists a $\bar{y} \in \mathbb{Z}^I_+$ such that $\bar{y}$ is the integral part of an optimal solution for any $d \in U$. Consequently, for some $t \in T$, z can be written as*

$$z(d) = c_I\bar{y} + \frac{c_t}{a_t}(d - a_I\bar{y}) \tag{14}$$

for all $d \in U$. Furthermore, for any $d \in U$, we have $d - a_I\bar{y} \geq 0$ if $t = t^+$ and $d - a_I\bar{y} \leq 0$ if $t = t^-$.

Proof. From Theorem 5 and our discussion above, we must either have $z(d) = \omega_{t^+}(d)$ or $z(d) = \omega_{t^-}(d)$ for all $d \in U$. Assume the former holds so that by Corollary 7, U is closed from the left. Let $(\bar{x}_I, \bar{x}_T)$ be an optimal solution for $b = \min\{d \mid d \in U\}$ with $z(b) = c_I\bar{x}_I + c_T\bar{x}_T$. Since z is linear on U with the slope η^C and passes through the point $(b, z(b))$, we can write $z(d) = g(\lfloor b \rfloor_{t^+}) + \eta^C(d - \lfloor b \rfloor_{t^+})$ for all $d \in U$. For any $q \in U$, we have $q \geq b$ and hence $\left(\bar{x}_I, \bar{x}_{t^+} + \frac{q-b}{a_{t^+}}, \bar{x}_{T\setminus t^+}\right)$ is a feasible solution for right-hand-side q. Observe that this is also an optimal solution since

$$
\begin{aligned}
c_I\bar{x}_I + c_{t^+}\left(\bar{x}_{t^+} + \frac{q-b}{a_{t^+}}\right) + c_{T\setminus t^+}x_{T\setminus t^+} &= c_I\bar{x}_I + c_T\bar{x}_T + c_{t^+}\left(\frac{q-b}{a_{t^+}}\right) \\
&= g(\lfloor b \rfloor_{t^+}) + \eta^C(b - \lfloor b \rfloor_{t^+}) + c_{t^+}\left(\frac{q-b}{a_{t^+}}\right) \\
&= g(\lfloor b \rfloor_{t^+}) + \eta^C(q - \lfloor b \rfloor_{t^+})
\end{aligned}
$$

The proof follows similar arguments for the case when z is determined by ω_{t^-}.

For the rest of the claim, note that if $t = t^+$ and $\bar{y}$ is the integral part of an optimal solution for d, then from our discussion in the proof of Proposition 4, $d - a\bar{y} \geq 0$ and $z(d) = c_I\bar{y} + \eta^C(d - a_I\bar{y})$. Similarly, if $t = t^-$, then $d - a\bar{y} \leq 0$ and $z(d) = c_I\bar{y} + \zeta^C(d - a_I\bar{y})$. □

Example 3 For the MILP instance (7), we have $T = \{4, 5\}$ and hence, x_6 is redundant. Furthermore, for the intervals $U_1 = \left[0, \frac{1}{2}\right], U_2 = \left[\frac{1}{2}, 1\right], U_3 = \left[1, \frac{7}{6}\right], U_4 = \left[\frac{7}{6}, \frac{3}{2}\right], \ldots$, we have $y^1 = (0\ 0\ 0), y^2 = (1\ 0\ 0), y^3 = (1\ 0\ 0), y^4 = (0\ 0\ 1), \ldots$ as the integral parts of the corresponding optimal solutions and therefore, plugging these values to (14), we obtain

$$
z(d) = \begin{cases} \ldots & \\ 2d & \text{if } d \in U_1 \\ -d + \frac{3}{2} & \text{if } d \in U_2 \\ 2d - \frac{3}{2} & \text{if } d \in U_3 \\ -d + 2 & \text{if } d \in U_4 \\ \ldots & \end{cases}
$$

□

As these results show, the continuous variables indexed by t^+ and t^-, and the associated quantities η^C and ζ^C are key components of the structure of the value function. It turns out that we can further state conditions on the continuity of z using these components.

Theorem 9

(i) $\eta^C < \infty$ *if and only if* z *is continuous from the right. Similarly,* $\zeta^C > -\infty$ *if and only if* z *is continuous from the left.*

(ii) Let $U \subset \mathbb{R}$, $V \subset \mathbb{R}$ *be the intervals that any two consecutive linear segments of* z *are defined on and let* $\alpha_U, \alpha_V \in \{\eta^C, \zeta^C\}$ *be the corresponding slopes. Then* $\alpha_U \neq \alpha_V$ *if and only if both* η^C *and* ζ^C *are finite if and only if* z *is continuous everywhere.*

Proof.

(i) Assume that $\eta^C < \infty$. Now, if $\zeta^C = -\infty$, then $z(d) = \omega_{t^+}(d)\ \forall d \in \mathbb{R}$ and from Theorem 6, z is continuous from the right. Otherwise, let $b \in \mathbb{R}$ be such that z is discontinuous at b. Then from Corollary 7, there exists a $\bar{y} \in \mathbb{Z}^I_+$ such that $b - a_I\bar{y} = 0$. In this case, both $\frac{Mb}{a_{t^+}}$ and $\frac{Mb}{a_{t^-}}$ are integral and therefore, $b = \lfloor b \rfloor_t, t \in T$ and $z(b) = \omega_{t^+}(b) = \omega_{t^-}(b)$. However, this is a contradiction, since ω_{t^+} is continuous from the right and ω_{t^-} is continuous from the left and therefore by lower-semicontinuity, z would then have to be continuous at b.

On the other hand, assume that z is continuous from the right. If $\eta^C = \infty$, then $z(d) = \omega_{t^-}(d)\ \forall d \in \mathbb{R}$, z is continuous from the left and z can only be continuous from the right when ω_{t^-} is linear over $\mathbb{R}$. By Proposition 3, this is possible only when $\eta^C = \zeta^C$, which is a contradiction.

Similar arguments can be made to show that the second part of the claim is also valid.

(ii) The proof follows directly from (i).

□

Theorem 9 states essentially that the continuity of z depends only on the finiteness of η^C and ζ^C and that when z is continuous, the slopes of the linear segments of z alternate between these two values. Consequently, pursuant to the results of Proposition 3, it is not hard to show that we must have $\eta^C = \infty \iff d^+_U = 0$ and $\zeta^C = -\infty \iff d^-_U = 0$. In other words, z overlaps with F_U in a right-neighborhood of the origin if and only if $\eta^C < \infty$ and similarly, in a left-neighborhood of the origin if and only if $\zeta^C > -\infty$.

Example 4 Consider the following MILP instance with a fixed right-hand side b

$$\begin{aligned} \min \quad & x_1 - \tfrac{3}{4}x_2 + \tfrac{3}{4}x_3 \\ \text{s.t} \quad & \tfrac{5}{4}x_1 - x_2 + \tfrac{1}{2}x_3 = b \quad \text{and} \\ & x_1, x_2 \in \mathbb{Z}_+, x_3 \in \mathbb{R}_+. \end{aligned} \tag{15}$$

As seen in Figure 2, the value function of problem (15) is continuous only from the right. For the intervals $U_1 = \left[0, \frac{1}{4}\right), U_2 = \left[\frac{1}{4}, \frac{1}{2}\right), U_3 = \left[\frac{1}{2}, \frac{3}{4}\right), U_4 = \left[\frac{3}{4}, 1\right), U_5 = \left[1, \frac{3}{2}\right), U_6 = \left[\frac{3}{2}, \frac{7}{4}\right), U_7 = \left[\frac{7}{4}, 2\right), U_8 = \{2\}$, we have $y^1 = (0\,0), y^2 = (1\,1), y^3 = (2\,2), y^4 = (3\,3), y^5 = (4\,4), y^6 = (2\,1), y^7 = (3\,2), y^8 = (4\,3)$ as the integral parts of the corresponding integral solutions. Note that z is discontinuous at $d_1 = 0, d_2 = \frac{1}{4}, d_3 = \frac{1}{2}, d_4 = 1, d_5 = \frac{5}{4}, d_6 = \frac{3}{2}, d_7 = \frac{7}{4}, d_8 = 2$ in the interval $[0, 2]$ and for each discontinuous point, we have $d_i - \left(\frac{5}{4}y^i_1 - y^i_2\right) = 0$. Furthermore, observe that $d^-_U = 0$, $d^+_U = \frac{1}{4}$ and each linear segment has the slope of $\eta^C = \frac{3}{2}$.

□

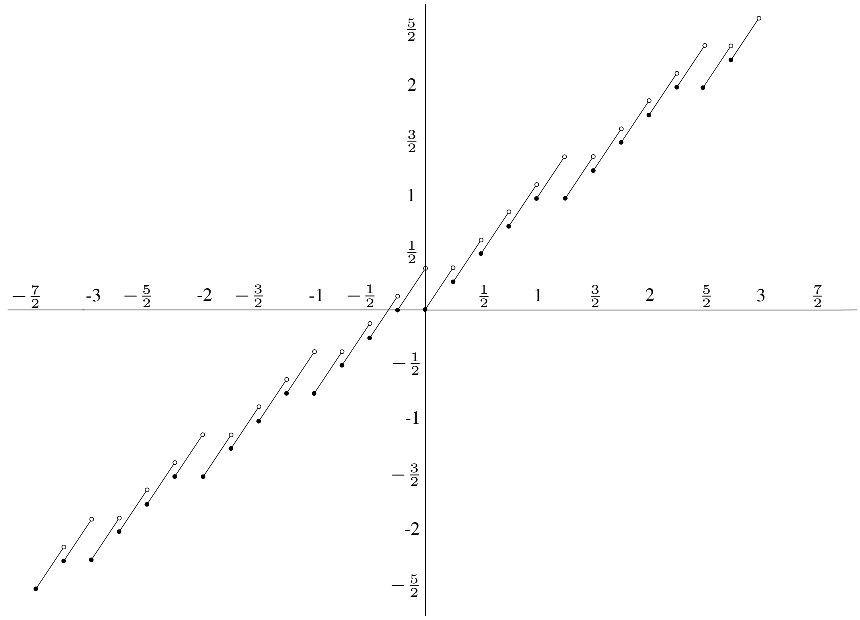

Figure 2: The value function of (15).

5 Maximal Subadditive Extension

Let a real-valued function f be subadditive on the interval $[0, h], h > 0$. The *maximal subadditive extension* of f to $\mathbb{R}_+$ is the function f_S defined for each $d \in \mathbb{R}_+$ by

$$f_S(d) = \begin{cases} f(d) & \text{if } \ d \in [0, h] \\ \inf\limits_{\mathcal{C} \in \mathcal{C}(d)} \sum\limits_{\rho \in \mathcal{C}} f(\rho) & \text{if } \ d > h \end{cases} , \tag{16}$$

where $\mathcal{C}(d)$ is the set of all finite collections $\{\rho_1, ..., \rho_R\}$ such that $\rho_i \in [0, h], i = 1, ..., R$ and $\sum_{i=1}^{R} \rho_i = d$. Each collection $\{\rho_1, ..., \rho_R\}$ is called an h*-partition* of d.

Theorem 10 *(Bruckner [1960]) Let $f : [0, h] \rightarrow \mathbb{R}$ be a subadditive function with $f(0) = 0$ and let f_S be defined as in (16). Then f_S is subadditive and if g is any other subadditive extension of f to $\mathbb{R}_+$, then $g \leq f_S$.*

Note that because of subadditivity, we can find the exact size of the h-partitions needed to be evaluated to get the value of $f_S(d)$ for any $d > h$.

Theorem 11 *For any $d > h$, let $k_d \geq 2$ be the integer such that $d \in \left(\frac{k_d}{2} h, \frac{k_d+1}{2} h\right]$. Then*

$$f_S(d) = \inf \left\{ \sum_{i=1}^{k_d} f(\rho_i) \mid \sum_{i=1}^{k_d} \rho_i = d, \rho_i \in [0, h], i = 1, ..., k_d \right\}. \tag{17}$$

Proof. Without loss of generality, consider an h-partition of d, $\{\rho_1, ..., \rho_{k_d+1}\}$, where k_d is defined as above. In this case, at least one pair of members of this partition must have a sum less than or equal to h, since otherwise, $k_d \sum_{i=1}^{k_d+1} \rho_i = k_d d > \frac{(k_d+1)k_d}{2} h$, which would be a contradiction to the fact that $d \leq \frac{k_d+1}{2} h$. Hence, there is another h-partition with the same sum and one less member, which means that any partition with more than k_d members is redundant.

□

Using Theorem 11, we can formulate a procedure to construct f_S recursively for all $d \in \mathbb{R}_+$ as follows:

i. $f_S(d) = f(d), d \in [0, h]$. Let $p := h$.

ii. For any $d \in \left(p, p + \frac{p}{2}\right]$, let

$$f_S(d) = \inf\{f_S(\rho_1) + f_S(\rho_2) \mid \rho_1 + \rho_2 = d, \rho_1, \rho_2 \in (0, p]\} \tag{RP}$$

Let $p := p + \frac{p}{2}$ and repeat this step.

Observe that this procedure extends the function over an interval that is the half of the previous interval so that we only need to consider p-partitions with size 2 for any $d \in \left(p, p + \frac{p}{2}\right]$ by Theorem 11.

One special case occurs when f_S has the property that $f_S(kh + d) = kf(h) + d$ for all $d \in [0, h]$ and all $k \in \mathbb{Z}_+$, i.e., the function "repeats" itself over intervals of size h. In particular, this property is observed whenever $f_S(h+d) = f(h)+f(d)$ for all $d \in (0, h]$ (Laatsch [1964]). The following result states necessary and sufficient conditions for this behavior.

Proposition 12 *(Barton and Laatsch [1966]) Let f be a real-valued subadditive function with $f(0) = 0$ and defined on $[0, h]$. Then $f_S(kh + d) = kf(h) + d$ for all $d \in [0, h]$ and all $k \in \mathbb{Z}_+$ if and only if for all $\rho_1 \in (0, h]$,*

$$f(\rho_1) \leq f(h + \rho_1 - \rho_2) - f(h) + f(\rho_2) \tag{18}$$

for all ρ_2 satisfying $\rho_1 \leq \rho_2 \leq h$.

Corollary 13 *Let f be a real-valued subadditive function with $f(0) = 0$ that is concave on $[0, h]$. Then $f_S(kh + d) = kf(h) + f(d)$ for all $d \in (0, h]$ and all $k \in \mathbb{Z}_+$.*

All the results above are also valid when extending a real-valued subadditive function f defined on the interval $[h, 0]$ for $h < 0$ to $\mathbb{R}_-$.

We now state one of our main results: if $F(d) = z(d)$ for all d in a certain neighborhood of the origin that we will define, then the maximal subadditive extension of F to all of $\mathbb{R}$ must be precisely the value function. Before we proceed, we first note that we can change the "inf" to "min" in (16) if the seed function is the value function itself.

Lemma 14 *Let the function* $f : [0,h] \to \mathbb{R}$ *be defined by* $f(d) = z(d)\ \forall d \in [0,h]$. *Then,*

$$f_S(d) = \begin{cases} z(d) & \text{if } d \in [0,h] \\ \min_{\mathcal{C} \in \mathcal{C}(d)} \sum_{\rho \in \mathcal{C}} z(\rho) & \text{if } d > h \end{cases} . \tag{19}$$

Proof. Assume that this is not the case and we have for $d > h$

$$\inf_{\mathcal{C} \in \mathcal{C}(d)} \sum_{\rho \in \mathcal{C}} z(\rho) < \sum_{\rho \in \mathcal{C}} z(\rho) \quad \forall \mathcal{C} \in \mathcal{C}(d) \tag{20}$$

Note that this is possible only if for some $\mathcal{C} \in \mathcal{C}(d)$ and $\rho \in \mathcal{C}$, we have

$$z(\rho) > \lim_{\sigma \to \rho^+} z(\sigma) \quad \text{or} \quad z(\rho) > \lim_{\sigma \to \rho^-} z(\sigma).$$

However, this is a contradiction to lower-semicontinuity of z given in Corollary 7. $\square$

Theorem 15 *Let* $d_r = \max\{a_i \mid i \in N\}$ *and* $d_l = \min\{a_i \mid i \in N\}$ *and let the functions* f_r *and* f_l *be the maximal subadditive extensions of z from the intervals* $[0, d_r]$ *and* $[d_l, 0]$ *to* $\mathbb{R}_+$ *and* $\mathbb{R}_-$, *respectively. If we define the function*

$$F(d) = \begin{cases} f_r(d) & d \in \mathbb{R}_+ \\ f_l(d) & d \in \mathbb{R}_- \end{cases} , \tag{21}$$

then $z = F$.

Proof. $z \le F$: For a given $d > d_r$, let $\{\rho_i\}_{i=1}^R, \rho_i \in [0, d_r]\ \forall i \in \{1, ..., R\}$ be a d_r-partition of d such that $f_r(d) = \sum_{i=1}^R z(\rho_i)$. Note that such a partition exists by Lemma 14. In addition, let $\{x^i\}_{i=1}^R$ be the collection of corresponding optimal solutions. Since $\sum_{i=1}^R x^i$ is a feasible solution to MILP (P) with right-hand side d, then $F(d) = f_r(d) = \sum_{i=1}^R z(\rho_i) \ge z(d)$. Hence, $F(d) \ge z(d)\ \forall d \ge 0$. Similarly, one can also show that $F(d) \ge z(d)\ \forall d < d_l$.

$z \ge F$: Using MILP duality, we show that F is dual feasible.

i. F *is subadditive*: It is enough to show that F is subadditive in $\left[\frac{3d_l}{2}, \frac{3dr}{2}\right]$ due to recursive procedure (RP). Let $d_1 \in \left[\frac{3}{2}d_l, 0\right]$ and $d_2 \in \left[0, \frac{3d_r}{2}\right]$ be given. From the construction of f_l, we know that $f_l(d_1)$ can be determined by a d_l-partition $\{\rho_i^1\}_{i=1}^R$ of d_1 which has a size of at most 2 and similarly, $f_r(d_2)$ can be determined by a d_r-partition $\{\rho_i^2\}_{i=1}^R$ of d_2 also with a size of at most 2. Then

$$\begin{aligned} F(d_1) + F(d_2) &= \sum_{i=1}^2 f_l(\rho_i^1) + \sum_{i=1}^2 f_r(\rho_i^2) \\ &= \sum_{i=1}^2 z(\rho_i^1) + z(\rho_i^2) \\ &\ge \sum_{i=1}^2 z(\rho_i^1 + \rho_i^2) , \end{aligned}$$

where the last inequality follows from the fact that z is subadditive in $[d_l, d_r]$ and pair sums $\rho_i^1 + \rho_i^2 \in [d_l, d_r], i = 1, 2$. If both these pair sums are either in $[d_l, 0]$ or in $[0, d_r]$, then we are done, since in this case, the subadditivity of f_l or f_r can be invoked directly to show that $\sum_{i=1}^{2} z(\rho_i^1 + \rho_i^2) \geq F(d_1 + d_2)$. Otherwise, we have at least one pair sum in each of these intervals. Without loss of generality, let $\rho_1^1 + \rho_1^2 \in [d_l, 0]$ and $\rho_2^1 + \rho_2^2 \in [0, d_r]$. Then we have

$$\sum_{i=1}^{2} z\left(\rho_i^1 + \rho_i^2\right) \quad \geq \quad z\left(\sum_{i=1}^{2} \rho_i^1 + \rho_i^2\right) = F(d_1 + d_2)$$

since $\sum_{i=1}^{2} \rho_i^1 + \rho_i^2 \in [d_l, d_r]$ and z is subadditive in this interval.

ii. $F(a_j) \leq c_j \; \forall j \in I$: Since the range $[d_l, d_r]$ is chosen to include all $a_j, j \in I$, then $F(a_j) = z(a_j) \leq c_j \; \forall j \in I$.

iii. $\bar{F}(a_j) = \limsup_{\delta \to 0^+} \frac{F(\delta a_j)}{\delta} \leq c_j \; \forall j \in C$: For any $j \in C$ and sufficiently small values of $\delta > 0$, we have $F(\delta a_j) = z(\delta a_j) \leq \delta c_j$. This implies that, $\bar{F}(a_j) \leq c_j \; \forall j \in C$.

Also note that $F(0) = z(0) = 0$ due to our initial assumptions and thus, F is a feasible dual function and from MILP duality, $z \geq F$. □

6 Evaluating the Value Function

Theorem 15 states essentially that z can be encoded simply by its two slopes and the set of its breakpoints in the interval $[d_l, d_r]$. Now that we can formulate z as in Theorem 15, an obvious question is whether we can algorithmically extract the value $z(d)$ for a given $d \in \mathbb{R} \backslash [d_l, d_r]$ from this encoding. The following theorem states that, for a special case, this can be done easily.

Theorem 16 *Let d_l, d_r be defined as in Theorem 15. If z is concave in $[0, d_r]$, then for any $d \in \mathbb{R}_+$*

$$z(d) = kz(d_r) + z(d - kd_r), \qquad kd_r \leq d < (k+1)d_r \; k \in \mathbb{Z}_+ \tag{22}$$

Similarly, if z is concave in $[d_l, 0]$, then for any $d \in \mathbb{R}_-$

$$z(d) = kz(d_l) + z(d - kd_l), \qquad (k+1)d_l < d \leq kd_l \; k \in \mathbb{Z}_+ \tag{23}$$

Proof. The proof follows from Corollary 13 and Theorem 15. □

Example 5 For the MILP instance (7), z is concave in $\left[\frac{-3}{2}, 0\right]$ and therefore, for any $d \in \mathbb{R}_-$, $z(d)$ can be written as in (23) which simply requires z to *repeat* itself over the intervals of size $\frac{3}{2}$ (see Figure 1). □

In general, however, the question is whether there is a finite algorithm to compute $z(d)$ from the above encoding. Our discussion in the rest of this section shows that indeed there is.

Without loss of generality, we will only consider the interval $\mathbb{R}_+ \backslash [0, d_r]$. Extending z from the interval $[d_l, 0]$ to $\mathbb{R}_-$ can be accomplished similarly.

Definitions

- Let $U_i, i \in \{1, \ldots, s\}$ be the maximal intervals defining the linear segments of z on $[0, d_r]$.
- For each $i \in \{1, \ldots, s-1\}$, let $d^i = \sup\{d \mid d \in U_i\} = \inf\{d \mid d \in U_{i+1}\}$. Setting $d^0 = 0, d^s = d_r$, let $\Phi \equiv \{d^0, \ldots, d^s\}$. We will call the set Φ the *break points* of z on the interval $[0, d_r]$.
- Let $\Psi \equiv \{d^i \mid \alpha^i \leq \alpha^{i+1}, i \in \{1, \ldots, s-1\}\} \cup \{0, d_r\}$ where each α^i is defined as in Theorem 9. We will call the set Ψ the *lower break points* of z on the interval $[0, d_r]$. Observe that $\Psi \equiv \Phi$ if z is not continuous everywhere.
- For any $d \in \mathbb{R}_+ \backslash [0, d_r]$, a d_r-partition $\mathcal{C} \in \mathcal{C}(d) \equiv \{\rho_1, \ldots, \rho_{k_d}\}$ is called an optimal partition if $z(d) = \sum_{i=1}^{k_d} z(\rho_i)$ where $k_d \geq 2$ is the integer such that $d \in \left(\frac{k_d}{2} d_r, \frac{k_d+1}{2} d_r\right]$. Note that such a partition exists by Theorem 11.

Theorem 17 *For any $d \in \mathbb{R}_+ \backslash [0, d_r]$, there is an optimal d_r-partition $\mathcal{C} \in \mathcal{C}(d)$ such that $|\mathcal{C} \backslash \Phi| \leq 1$.*

Proof. Let $\mathcal{C} \equiv \{\rho_1, \ldots, \rho_{k_d}\}$ be an optimal d_r-partition for d. If $|\mathcal{C} \backslash \Phi| \leq 1$, then we are done. Otherwise, there is a pair $\{\rho_l, \rho_r\} \in \mathcal{C} \backslash \Phi$. Without loss of generality, let $\rho_l \leq \rho_r$, $\rho_l \in U_l$ and $\rho_r \in U_r$.

- *z is continuous everywhere*: In this case, both η^C and ζ^C are finite and $\alpha^l, \alpha^r \in \{\eta^C, \zeta^C\}$.
  - $\alpha^l = \alpha^r$: Let $\theta = \rho_l - \min\{d \mid d \in U_l\} = \rho_l - d^{l-1}$ and $\lambda = \max\{d \mid d \in U_r\} - \rho_r = d^r - \rho_r$. Setting $\delta = \min\{\theta, \lambda\}$ and $S \equiv \{\rho_l - \delta, \rho_r + \delta\}$, clearly, $S \cup \mathcal{C} \backslash \{\rho_l, \rho_r\}$ is another optimal d_r-partition for d and $|S \cup \mathcal{C} \backslash \{\rho_l, \rho_r\} \cap \Phi| > |\mathcal{C} \cap \Phi|$.
  - $\alpha^l > \alpha^r$: Then, $z(\rho_l - \delta) + z(\rho_r + \delta) < z(\rho_l) + z(\rho_r)$ and therefore, $\mathcal{C}$ cannot be an optimal partition for d.
  - $\alpha^l < \alpha^r$: If we let $\theta = \max\{d \mid d \in U_l\} - \rho_l = d^l - \rho_l$, $\lambda = \rho_r - \min\{d \mid d \in U_r\} = \rho_r - d^{r-1}$, and $\delta = \min\{\theta, \lambda\}$, then $z(\rho_l + \delta) + z(\rho_r - \delta) < z(\rho_l) + z(\rho_r)$ and again, this is a contradiction to the optimality of the partition $\mathcal{C}$.
- *z is not continuous everywhere*: In this case, only one of η^C and ζ^C is finite and either $\alpha^l = \alpha^r = \eta^C < \infty$ or $\alpha^l = \alpha^r = \zeta^C > -\infty$.
  - $\eta^C < \infty$: From Theorem 9, we know that each U_i $i \in \{1, \ldots, s\}$ is closed from the left and therefore $d^i \in U_{i+1}$ $i \in \{0, \ldots, s-1\}$.

Now, let $\theta = \sup\{d \mid d \in U_l\} - \rho_l = d^l - \rho_l$, $\lambda = \rho_r - \min\{d \mid d \in U_r\} = \rho_r - d^{r-1}$ and $\delta = \min\{\theta, \lambda\}$. If $\theta > \lambda$, then, with $S \equiv \{\rho_l + \delta, \rho_r - \delta\}$, $S \cup \mathcal{C} \backslash \{\rho_l, \rho_r\}$ is another optimal d_r-partition for d and $|S \cup \mathcal{C} \backslash \{\rho_l, \rho_r\} \cap \Phi| > |\mathcal{C} \cap \Phi|$. If $\theta \leq \lambda$, then $\mathcal{C}$ can not be an optimal partition for d since $\rho_l + \theta = d^l$ and from lower-semicontinuity of z, $z(\rho_l + \theta) + z(\rho_r - \theta) < z(\rho_l) + z(\rho_r)$.

- $\zeta^C > -\infty$: With similar arguments, one can show that the claim is still valid for this case.

Note that this procedure updates the optimal partition so that the number of common members of the new optimal partition and Φ increases at least by 1. Therefore, applying the procedure iteratively would yield an optimal partition $\bar{\mathcal{C}} \in \mathcal{C}(d)$, $\bar{\mathcal{C}} \equiv \{\bar{\rho}_1, \ldots, \bar{\rho}_{k_d}\}$ so that $|\bar{\mathcal{C}} \backslash \Phi| \leq 1$.

□

Example 6 Consider the MILP instance (15). We have $d_r = \frac{5}{4}$ and $\Phi = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}\right\}$. For $b = \frac{17}{8}$, $k_b = 3$ and $\mathcal{C} = \left\{\frac{1}{8}, \frac{3}{4}, \frac{5}{4}\right\}$ is an optimal d_r-partition with $|\mathcal{C} \backslash \Phi| = 1$.

□

In fact, we can further restrict the search space for the continuous case by only considering the *lower break points*. Note that this would be expected, considering Corollary 13.

Corollary 18 *If both η^C and ζ^C are finite, then for any $d \in \mathbb{R}_+ \backslash [0, d_r]$ there is an optimal d_r-partition $\mathcal{C} \in \mathcal{C}(d)$ such that $|\mathcal{C} \backslash \Psi| \leq 1$.*

Proof. Assume that $\mathcal{C} \equiv \{\rho_1, \ldots, \rho_{k_d}\}$ is an optimal d_r-partition for d such that ρ_r, $\rho_l \in \mathcal{C} \backslash \Psi$. From Theorem 17, we know that at least one of ρ_r, ρ_l is in Φ.

- $\rho_r, \rho_l \in \Phi$: Since z is continuous everywhere, then $\rho_r \in U_r \cap U_{r+1}$ with $\alpha^r > \alpha^{r+1}$ and $\rho_l \in U_l \cap U_{l+1}$ with $\alpha^l > \alpha^{l+1}$. Let $\lambda = \rho_r - \min\{d \mid d \in U_r\} = \rho_r - d^{r-1}$, $\theta = \max\{d \mid d \in U_{l+1}\} - \rho_l = d^{l+1} - \rho_l$ and $\delta = \min\{\theta, \lambda\}$. Clearly, $\mathcal{C}$ cannot be an optimal partition since for any ϵ with $0 < \epsilon < \delta$, $\{\rho_l + \epsilon, \rho_r - \epsilon\} \cup \mathcal{C} \backslash \{\rho_l, \rho_r\}$ is a d_r-partition for d with $z(\rho_l + \epsilon) + z(\rho_r - \epsilon) < z(\rho_l) + z(\rho_r)$.

- $\rho_r \in \Phi$ and $\rho_l \notin \Phi$: Similarly, assume that $\rho_r \in U_r \cap U_{r+1}$ with $\alpha^r > \alpha^{r+1}$ and $\rho_l \in U_l$.
  - $\alpha^l = \eta^C$: Let $\theta = \rho_l - \min\{d \mid d \in U_l\} = \rho_l - d^{l-1}$, $\lambda = \max\{d \mid d \in U_{r+1}\} - \rho_r = d^{r+1} - \rho_r$, $\delta = \min\{\theta, \lambda\}$. Then, $\mathcal{C}$ can not be an optimal partition for d since for any ϵ with $0 < \epsilon < \delta$, $\{\rho_l - \epsilon, \rho_r + \epsilon\} \cup \mathcal{C} \backslash \{\rho_l, \rho_r\}$ is a d_r-partition for d with $z(\rho_l - \epsilon) + z(\rho_r + \epsilon) < z(\rho_l) + z(\rho_r)$.
  - $\alpha^l = \zeta^C$: Similarly, let $\theta = \max\{d \mid d \in U_l\} - \rho_l$, $\lambda = \rho_r - \min\{d \mid d \in U_r\}$, $\delta = \min\{\theta, \lambda\}$. Then again, $\mathcal{C}$ can not be an optimal partition for d since for any ϵ with $0 < \epsilon < \delta$, $\{\rho_l + \epsilon, \rho_r - \epsilon\} \cup \mathcal{C} \backslash \{\rho_l, \rho_r\}$ is a d_r-partition for d and $z(\rho_l + \epsilon) + z(\rho_r - \epsilon) < z(\rho_l) + z(\rho_r)$.

Therefore, and from Theorem 17, we conclude that $|\mathcal{C} \backslash \Psi| \leq 1$.

□

Example 7 Consider the MILP instance (7). For the interval $\left[0, \frac{3}{2}\right]$, we have $\Psi = \{0, 1, \frac{3}{2}\}$. For $b = \frac{25}{8}$, $\mathcal{C} = \{\frac{1}{8}, \frac{3}{2}, \frac{3}{2}\}$ is an optimal d_r-partition with $|\mathcal{C}\backslash\Psi|$ = 1.

□

From Theorem 17 and Corollary 18, we conclude that we need only to search a finite space of d_r-partitions to obtain the value of $z(d)$ for any $d \in \mathbb{R}_+\backslash[0, d_r]$. In particular, we only need to consider the collection set

$$\Lambda(d) \equiv \{\mathcal{C} \mid \mathcal{C} \in \mathcal{C}(d), |\mathcal{C}| = k_d, |\mathcal{C}\backslash\Psi| \leq 1\}\,, \tag{24}$$

which can be written equivalently as

$$\Lambda(d) \equiv \left\{\mathcal{H} \cup \{\mu\} \mid \mathcal{H} \in \mathcal{C}(d-\mu), |\mathcal{H}| = k_d - 1, \mathcal{H} \subseteq \Psi, \sum_{\rho\in\mathcal{H}} \rho + \mu = d, \mu \in [0, d_r]\right\}. \tag{25}$$

In other words,

$$z(d) = \min_{\mathcal{C}\in\Lambda(d)} \sum_{\rho\in\mathcal{C}} z(\rho). \tag{26}$$

Observe that the set $\Lambda(d)$ is finite, since Ψ is finite due to the fact that z has finitely many linear segments on $[0, d_r]$ and since for each $\mathcal{H} \subseteq \Psi$ with $|\mathcal{H}| = k_d - 1$, $\mu \in [0, d_r]$ is uniquely determined if $\mathcal{H} \cup \mu \in \Lambda(d)$. In particular, first, from our discussion in the proof of Theorem 6,

$$|\Psi\backslash\{0, d_r\}| \leq \min_{t\in T} \left\{ M \left| \frac{d_r}{a_t} \right| \right\} \tag{27}$$

where M is defined as before. The relation (27) is due to the fact that for any $d \in (0, d_r)$, w_t is linear over the interval $\left[\lfloor d \rfloor_t, \lfloor d \rfloor_t + \frac{a_t}{M}\right)$ if $t = t^+$ and $\left(\lfloor d \rfloor_t + \frac{a_t}{M}, \lfloor d \rfloor_t\right]$ if $t = t^-$ and $d \in \Psi$ has to be a break point of both w_{t^+} and w_{t^-} if η^C and ζ^C are finite. Consequently, we can get a bound on the cardinality of the set $\Lambda(d)$ considering the special collections of the members of Ψ of size $k_d - 1$. Since in such a collection, each member can be chosen more than once and the order of the members do not matter, we have

$$\Lambda(d) \leq \begin{pmatrix} |\Psi| + k_d - 2 \\ k_d - 1 \end{pmatrix}. \tag{28}$$

With these observations, we can reformulate the recursive procedure (RP) in a more practical way. Setting $\Psi(p)$ to the set of the lower break points of z in the interval $(0, p]$ $p \in \mathbb{R}_+$, we have

i. Let $p := d_r$.

ii. For any $d \in \left(p, p + \frac{p}{2}\right]$,

$$z(d) = \min\{z(\rho_1) + z(\rho_2) \mid \rho_1 + \rho_2 = d, \rho_1 \in \Psi(p), \rho_2 \in (0, p]\} \tag{29}$$

Let $p := p + \frac{p}{2}$ and repeat this step.

In fact, we can also write (29) as follows:

$$z(d) = \min_j g^j(d) \; \forall d \in \left(p, p + \frac{p}{2}\right] \tag{30}$$

where, for each $d^j \in \Psi(p)$, the functions $g^j : \left[0, p + \frac{p}{2}\right] \to \mathbb{R} \cup \{\infty\}$ are defined as

$$g^j(d) = \begin{cases} z(d) & \text{if } \; d \leq d^j, \\ z(d^j) + z(d - d^j) & \text{if } \; d^j < d \leq p + d^j, \\ \infty & \text{otherwise.} \end{cases} \tag{31}$$

Because of subadditivity, we can then write

$$z(d) = \min_j g^j(d) \; \forall d \in \left(0, p + \frac{p}{2}\right]. \tag{32}$$

Example 8 Consider the MILP instance (7). For the left side of the origin, we have only one lower break point $-\frac{3}{2}$ (other than origin) and therefore z repeats itself over intervals of size $-\frac{3}{2}$. Note that this result is parallel to the conclusion of Theorem 16. For the right side of the origin, we have $\Psi(\frac{3}{2}) = \{1, \frac{3}{2}\}$. Then,

$$g^1(d) = \begin{cases} z(d) & 0 \leq d \leq 1, \\ \frac{1}{2} + z(d-1) & 1 < d \leq \frac{9}{4}. \end{cases} \qquad g^2(d) = \begin{cases} z(d) & 0 \leq d \leq \frac{3}{2}, \\ \frac{1}{2} + z(d - \frac{3}{2}) & \frac{3}{2} < d \leq \frac{9}{4}. \end{cases}$$

and consequently (see Figure 3),

$$z(d) = \min\{g^1(d), g^2(d)\} \;, \; d \in \left[0, \frac{9}{4}\right].$$

Once we have obtained z in $\left[0, \frac{9}{4}\right]$, we can apply the same procedure in order to extend z to $\left[0, \frac{27}{8}\right]$ (see Figure 4).

□

7 Conclusion

In this paper, we presented some theoretical results related to the structure of the value function of a MILP with a single constraint. It is our hope that this work will provide the foundation for developing practical methods for approximating the value function of a general MILP and will lead to practical methods for sensitivity analysis and warm starting. It is clear that a number of results presented herein can be extended to more general cases, but such extension seems unlikely to yield any practical methods. In the case of a MILP with a single constraint, however, it does seem that practical methods may be possible, since the enumeration required to evaluate the value function for a new right-hand side is reasonable in cases where the coefficients and right-hand side are small in absolute value. Computational results with such enumeration methods will be reported in a separate paper.

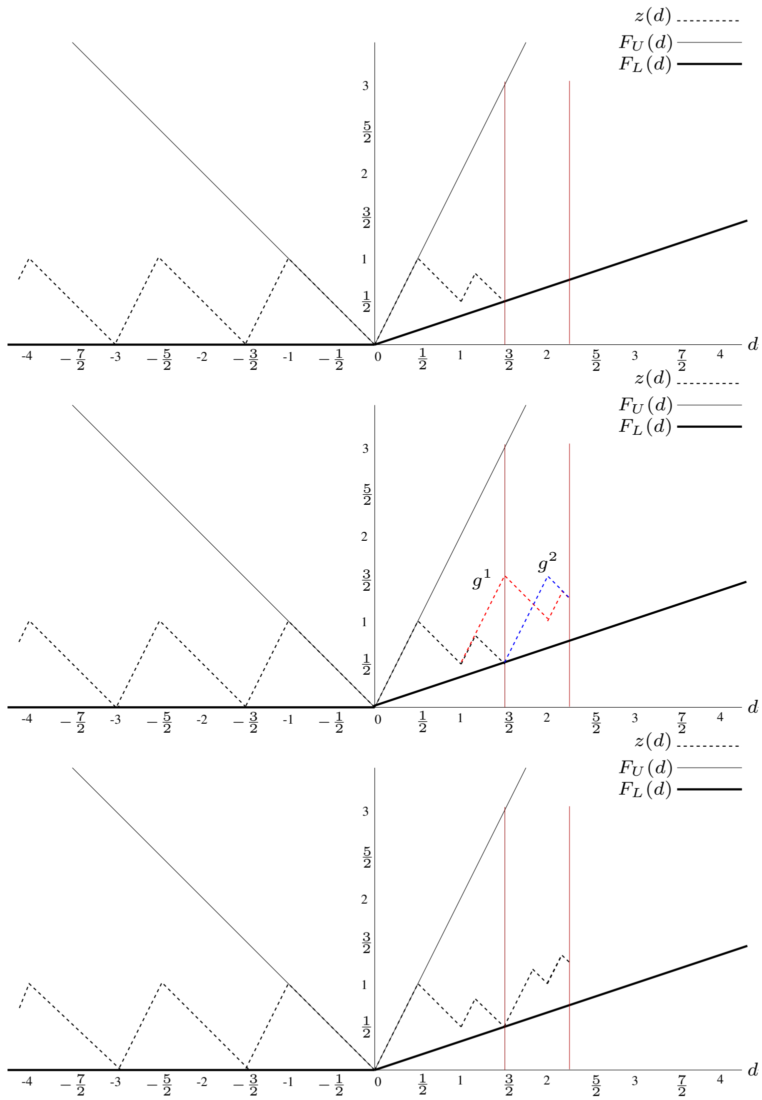

Figure 3: Extending the value function of (7) from $\left[0, \frac{3}{2}\right]$ to $\left[0, \frac{9}{4}\right]$.

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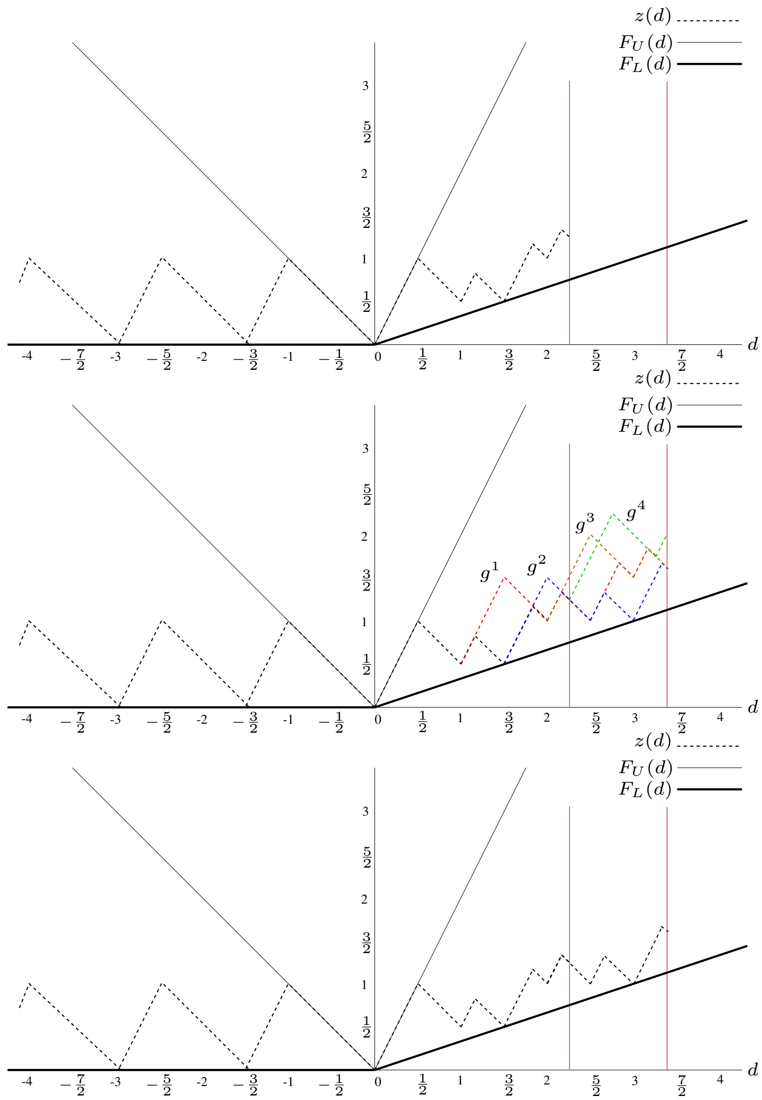

Figure 4: Extending the value function of (7) from $\left[0, \frac{9}{4}\right]$ to $\left[0, \frac{27}{8}\right]$.

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