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On the Complexity of Inverse Mixed Integer Linear Optimization

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Abstract

Inverse optimization is the problem of determining the values of missing input parameters that are closest to given estimates and that will make a given target solution optimal. This study is concerned with inverse mixed integer linear optimization problems (MILPs) in which the missing parameters are objective function coefficients. This class generalizes the class studied by Ahuja and Orlin [2001], who showed that inverse (continuous) linear optimization problems can be solved in polynomial time under mild conditions. We extend their result to the discrete case and show that the decision version of the inverse MILP is coNP–complete, while the optimal value verification problem is D^{P}–complete. We describe a cutting plane algorithm for solving inverse MILPs and show that there is a close relationship between the inverse problem and the well-known separation problem in both a complexity and an algorithmic sense.

Keywords: Inverse optimization, mixed integer linear optimization, computational complexity, polynomial hierarchy

1 Introduction

Optimization problems arise in many fields and the literature abounds with techniques for solving various classes of such problems. In general the goal of optimization is to determine a member of a given feasible set (an *optimal solution*) that minimizes the value of a given *objective function*. The feasible set is typically described as the points in a vector space satisfying a given set of equations, inequalities, and disjunctions (the latter are usually in the form of a requirement that the value of a certain element of the solution take on an integral value).

An *inverse optimization problem*, in contrast, is a related problem in which the description of the original optimization problem, which we refer to as the *forward problem*, is not complete (some parameters are missing or cannot be observed), but a full or partial solution *can* be observed. The goal is to determine values for the missing parameters with respect to which the given solution would be optimal for the resulting problem. Estimates for the missing parameters may be given, in which case the goal is to produce a set of parameters that is as "close" to the given estimates as possible.

1.1 Formal Definitions

The optimization problem of interest in this paper is the *mixed integer linear optimization problem*

$$\min_{x \in \mathcal{S}} d^\top x \tag{MILP}$$

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where $d \in \mathbb{Q}^n$ and

$$\mathcal{S} = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\} \cap (\mathbb{Z}^r \times \mathbb{R}^{n-r}).$$

for $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$.

One can define a number of different inverse problems associated with (MILP), depending on what parts of the description (A, b, d) are unknown. Here, we study the case in which the objective function d is unknown, but we are given A and b, as well as an estimate $c \in \mathbb{Q}^n$ of the true objective d and a solution $x^0 \in \mathbb{Q}^n$. Formalizing the statement of this problem requires a careful attention to details that we now highlight by discussing several candidate formulations for this inverse problem.

We first consider the following formulation of the inverse problem as a semi-infinite optimization problem:

$$\begin{aligned} \min \quad & \|c - d\| \\ \text{s.t.} \quad & d^\top x^0 \leq d^\top x \quad \forall x \in \mathcal{S}, \end{aligned} \tag{INVMILP}$$

where $\|\cdot\|$ can be any norm. In (INVMILP), d is the unspecified vector to be determined (and is thus a vector of variables here rather than being fixed), while $c \in \mathbb{Q}^n$ is the estimate or target value. Note that in (INVMILP), if we instead x^0 vary, replacing it with a variable x, and interpret d as a fixed objective function, replacing $\|c - d\|$ with the objective $d^\top x$ of the forward problem, we get a reformulation of the forward problem (MILP) itself.

Problem (INVMILP) can also be re-formulated as a conic problem. In terms of the conic sets

$$\begin{aligned} &\mathcal{K}(y) = \{\alpha d \in \mathbb{R}^n : \|c - d\| \leq y, \alpha > 0, \alpha \in \mathbb{R}\} \text{ and} \\ &\mathcal{D} = \{d \in \mathbb{R}^n : d^\top (x^0 - x) \leq 0 \; \forall x \in \mathcal{S}\}, \end{aligned}$$

(INVMILP) can be reformulated as

$$\min_{d \in \mathcal{K}(y) \cap \mathcal{D}} y. \tag{INVMILP-C}$$

The set $\mathcal{D}$ can be interpreted either as the set of objective function vectors for which x^0 is preferred over all points in $\mathcal{S}$ or, alternatively, as the set of all hyperplanes containing x^0 that define inequalities valid for $\mathcal{S}$. The latter interpretation leads to a third formulation in terms of the so-called 1-polar. For a given polyhedron $\mathcal{P}$, e.g., $\text{conv}(\mathcal{S})$, the 1-polar is defined as

$$\mathcal{P}^1 = \{\pi \in \mathbb{R}^n \mid \pi^\top x \geq 1, \forall x \in \mathcal{P}\},$$

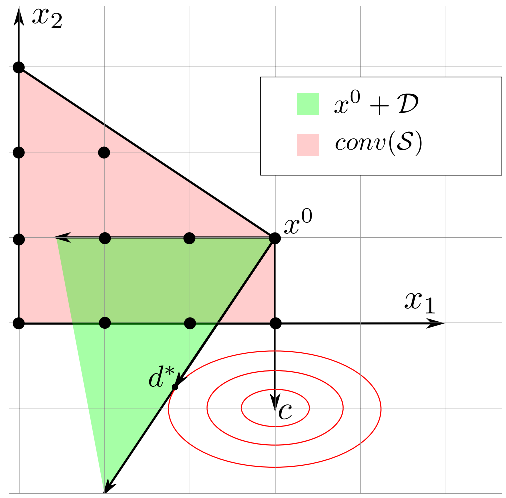
assuming that $\mathcal{P}$ is a polytope. When $\mathcal{P}$ is full-dimensional and $0 \in \mathcal{P}$ (this latter requirement is without loss of generality by translating $\mathcal{P}$), the 1-polar is the normalized set of all inequalities valid for $\mathcal{P}$ (see [Schrijver, 1986] for formal definitions). Under these assumptions, (INVMILP) can also be reformulated as

$$\begin{aligned} \min \quad & \|c - d\| \\ \text{s.t.} \quad & \pi \in \mathcal{P}^1 \\ & \pi^\top x^0 \leq 1 \\ & d = \alpha \pi \\ & \alpha \in \mathbb{R}_+. \end{aligned} \tag{INVMILP-1P}$$

In formulation (INVMILP-1P), the constraint $d = \alpha\pi$ allows d to be scaled in order to improve the objective function value. We might also require $\|c\| = 1$ or normalize in some other way to avoid this scaling. The constraint $\pi^\top x^0 \leq 1$ ensures that d is feasible to (INVMILP). Observe also that relaxing the constraint $\pi^\top x^0 \leq 1$ yields a problem something like the classical separation problem, but with a different objective function. We revisit this idea in Section 2.

We have so far avoided an important point and that is what assumptions we make about the point x^0. On the one hand, the problem, as informally stated, can only have a solution if $x^0 \in \mathcal{S}$, since otherwise, x^0 cannot be optimal for any objective function. On the other hand, the formulations above can be interpreted

Figure 1: Two dimensional inverse MILP



whether or not $x^0 \in \mathcal{S}$. As a practical matter, this subtle point is not very important, since membership in $\mathcal{S}$ can be verified in a pre-processing step if necessary. However, in the context of formal complexity analysis, this point *is* important and we will return to it. For now, we do not assume $x^0 \in \mathcal{S}$, in which case d can be more accurately interpreted as specifying a valid inequality which is satisfied at equality by x^0.

In order to paint a complete picture, there is one other pathological case to be considered and that is when x^0 is in the (relative) interior of conv($\mathcal{S}$). In this case, any objective vector in the subspace orthogonal to the affine space containing conv($\mathcal{S}$) is feasible for the inverse problem, i.e., optimizes x^0. Define $c_{\mathcal{S}}$ as the projection of c onto the smallest affine space that contains $\mathcal{S}$. Define $c_{\mathcal{S}}^{\perp}$ as the projection of c onto the orthogonal subspace. Then $c = c_{\mathcal{S}} + c_{\mathcal{S}}^{\perp}$ and $c_{\mathcal{S}} \perp c_{\mathcal{S}}^{\perp}$. When conv($\mathcal{S}$) is full dimensional, then $c = c_{\mathcal{S}}$, $d^* = c_{\mathcal{S}}^{\perp} = 0$ and the optimal value to the inverse problem is $\|c_{\mathcal{S}}\| = \|c\|$. When c is in the orthogonal subspace, then $c = c_{\mathcal{S}}^{\perp}$, $d^* = c_{\mathcal{S}}^{\perp}$ and the optimal value to the inverse problem is 0.

When x^0 is in the (relative) interior of conv($\mathcal{S}$), the inverse problem reduces to that of finding the closest point to c in orthogonal subspace. The optimal value for d in this case is the projection of c onto the orthogonal subspace, i.e., $d^* = c_{\mathcal{S}}^{\perp}$. The optimal objective value of the inverse problem is $\|c_{\mathcal{S}}\|$. In general, if we make no assumption about x^0, the optimal value of the inverse problem is bounded by 0 from below and $\|c_{\mathcal{S}}|$ from above.

Figure 1 demonstrates the inverse MILP geometrically. $\mathcal{S}$ is a discrete set indicated by the black dots. The vector $c = (0, -2)$ and $x^0 = (3, 1)$. The convex hull of $\mathcal{S}$ and the cone $\mathcal{D}$ (translated to x^0) are shaded. The ellipsoids show the sets of points with a fixed distance to $x^0 + c$ for some given norm. The optimal objective function in this example is vector d^*, and the point indicated in the figure is $x^0 + d^*$.

1.2 Previous Work

There are a range of different flavors of inverse optimization problem. The inverse problem we investigate is to determine objective function coefficients that make a given solution optimal, but other flavors of inverse optimization include constructing a missing part of either the coefficient matrix or the right-hand side that makes a given solution optimal. Heuberger [2004] provides a detailed survey of inverse combinatorial optimization problems. In this paper, different types of inverse problems, including types for which the inverse problem seeks parameters other than objective function coefficients, are examined. A survey of solution procedures for specific combinatorial problems is provided, as well as a classification of the inverse problems that are common in the literature. According to this classification, the inverse problem we study in this paper is an *unconstrained*, *single feasible object*, and *unit weight norm* inverse problem. Our results can be straightforwardly extended to some related cases, such as multiple given solutions.

Cai et al. [1999] examines an inverse center location problem in which the aim is to construct part of the coefficient matrix, in this case the distances between nodes from a given optimal solution. It is shows that even though the center location problem is polynomially solvable, this particular inverse inverse problem is NP–hard. This is done by way of a polynomial transformation of the satisfiability problem to the decision version of the inverse center location problem. This analysis indicates that the problem of constructing part of the coefficient matrix is harder than the forward version of the problem.

Huang [2005] examines the inverse knapsack problem and inverse integer optimization problems. In this paper, a pseudo–polynomial algorithm for the inverse knapsack problem is presented. It is also shown that inverse integer optimization with a fixed number of constraints is pseudo–polynomial by transforming the inverse problem to a shortest path problem on a directed graph. When the number of constraints are fixed, this results a pseudo–polynomial algorithm for inverse integer optimization.

Schaefer [2009] studies general inverse integer optimization problems. Using super-additive duality, a polyhedral description of the set of all feasible objective functions is derived. This description has only continuous variables but an exponential number of constraints. A solution method using this polyhedral description is proposed. Finally, Wang [2009] suggests a cutting plane algorithm similar to the one suggested herein and presents computational results on several test problem with an implementation of this algorithm.

The case when the feasible set is an explicitly described polyhedron is well–studied by Ahuja and Orlin [2001]. In their study, they analyze the shortest path, assignment, minimum cut, and minimum cost flow problems under the l_1 and l_∞ norms in detail. They also conclude that inverse optimization problem is polynomially solvable when the forward problem is polynomially solvable. The present study aims to generalize the result of Ahuja and Orlin [2001] to the case when the forward problem is not necessarily polynomially solvable, as well as to make connections to other well-known problems.

In the remainder of the paper, we first introduce address the computational complexity of (INVMILP). As written, this is a semi-infinite program, but it is easy to see that we can replace the infinite set of constraints with a finite set corresponding to the extreme points of $\text{conv}(\mathcal{S})$. This still leaves us with what is ostensibly a non-linear objective function. We show in Section 2 that for the ℓ_∞ and ℓ_1 norms, this problem can be expressed as a standard *linear optimization problem* (LP), albeit one with an exponential number of constraints. The reformulation can be readily solved in practice using a standard cutting plane approach. On the other hand, we show in Section 3 that the formal complexity does not depend on the norm.

2 Algorithmic Approach to Inverse MILP

We now show how to formulate (INVMILP) explicitly for two common norms using standard techniques for linearization. The objective function of an inverse MILP under the l_1 norm can be linearized by the introduction of variable vector θ, and associated constraints as follows,

$$\begin{aligned}
z_1^{-1} = \quad & \min \quad y \\
\text{s.t.} \quad & y = \sum_{i=1}^{n} \theta_i \\
& c_i - d_i \le \theta_i \quad \forall i \in \{1, 2, \ldots, n\} \\
& d_i - c_i \le \theta_i \quad \forall i \in \{1, 2, \ldots, n\} \\
& d^\top x^0 \le d^\top x \quad \forall x \in \mathcal{S}.
\end{aligned} \tag{INVMILP-1}$$

The objective function of inverse MILP under l_∞ norm can be linearized by the introduction of variable y and two sets of constraint sets as follows,

$$\begin{aligned}
z_\infty^{-1} = \quad & \min \quad y \\
\text{s.t.} \quad & c_i - d_i \le y \quad \forall i \in \{1, 2, \ldots, n\} \\
& d_i - c_i \le y \quad \forall i \in \{1, 2, \ldots, n\} \\
& d^\top x^0 \le d^\top x \quad \forall x \in \mathcal{S}.
\end{aligned} \tag{INVMILP-∞}$$

This formulation is a continuous problem, but is a semi-infinite program when written in the form above, as mentioned earlier.

To obtain a finite problem, we can limit the set of constraints to only those involving the finite set of extreme points and rays of $\text{conv}(\mathcal{S})$. Although this yields a finite formulation, the number of extreme points and rays may still be very large and it is not practical to write this formulation explicitly via a priori enumeration.

A better approach is to use a separation–optimization procedure and generate these inequalities dynamically, as suggested by Wang [2009]. This is a natural application of the separation–optimization procedure described in, e.g., Grötschel et al. [1993]. Although this approach has already been described in the literature, our purpose in describing it again here is is to make the connection to a similar existing algorithm for solving the standard separation problem, which provides the intuition behind the complexity results to be presented in Section 3.

The form of (INVMILP-1) and (INVMILP-∞) makes it clear that these two formulations, although of exponential size, can be solved by a standard cutting plane approach. We describe such a cutting plane algorithm for the case of the l_∞ norm (formulation (INVMILP-∞)) and note that a similar algorithm can be applied to the model (INVMILP-1) for the case of the l_1 norm.

First, let us define two parametric problems, P_k and $InvP_k$, as follows:

$$\min_{x\in\mathcal{S}} d^{k\top} x \tag{P_k}$$

$$\tag{1}$$

$$\begin{aligned} \min \quad & y \\ \text{s.t.} \quad & c_i - d_i \le y && \forall i \in \{1,2,\dots,n\} \\ & d_i - c_i \le y && \forall i \in \{1,2,\dots,n\} \\ & d^\top x^0 \le d^\top x && \forall x \in \mathcal{E}^k. \end{aligned} \tag{$InvP_k$}$$

where $\mathcal{E}^k$ is the set of solutions found by solving $P_1,\dots,P_{k-1}$. Note that (P_k) is an MILP with the same feasible region as the original forward problem (MILP), but with objective function d^k. This is precisely the problem of separating d^k from the feasible region of (INVMILP). $(InvP_k)$ is the relaxation of MILP (INVMILP-∞) considering only valid inequalities that correspond to solutions to the forward problem contained in $\mathcal{E}^k$.

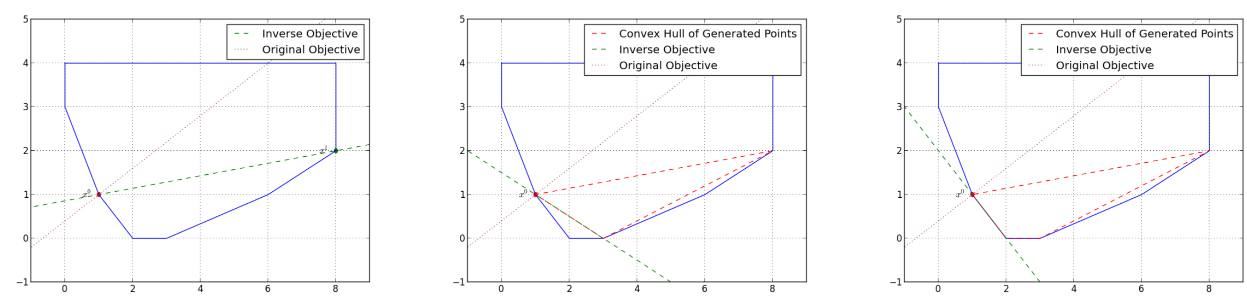
The overall procedure is given in Algorithm 1. In this algorithm, we solve an instance of the forward problem in each iteration in order to generate a cut. The algorithm stops when the current d^k is feasible. When (P_k) is unbounded, then $d = 0$ is an optimal solution, since this shows that only $d = 0$ satisfies $d^\top(x^0 - x) \le 0$ for all x in $\mathcal{S}$.

Before illustrating with a small example, we would like to again point out the close relationship of the inverse problem and the separation problem. First, note that another way of interpreting $(InvP_k)$ is as the problem of generating an inequality valid for $\text{conv}\,\mathcal{E}^k$ and for which the associated hyperplane, $\{x \in \mathbb{R}^n \mid {d^k}^\top x = {d^k}^\top x^0\}$, contains x^0. In this case, (P_k) can then be interpreted as the problem of determining whether there is an $x^k \in \mathcal{S}$, such that ${d^k}^\top x^k < {d^k}^\top x^0$, i.e., is violated by the associated valid inequality. This shows both that the inequality is not valid for $\text{conv}(\mathcal{S})$ and that d^k is not feasible for (INVMILP). Figure 2 illustrates how the algorithm might proceed for an example where the set $\mathcal{S}$ is the integer points inside the blue polyhedron.

Algorithm 1 can be easily modified to solve the generic separation problem for $\text{conv}(\mathcal{S})$ by interpreting x^0 as the point to be separated and replacing the objective function (and associated auxiliary constraints) of $(InvP_k)$ with one measuring the degree of violation of x^0. In this case, $(InvP_k)$ can be interpreted as the problem of separating x^0 from $\text{conv}\,\mathcal{E}^k$. Roughly, the dual of $(InvP_k)$ is to determine whether x^0 can be expressed as a convex combination of the members of $\mathcal{E}^k$. If not, then the proof is a separating hyperplane, which is an inequality valid for $\text{conv}\,\mathcal{E}^k$. As in the inverse case, (P_k) is interpreted as the problem of determining whether there is an $x^k \in \mathcal{S}$ that is violated by the associated valid inequality. The generated

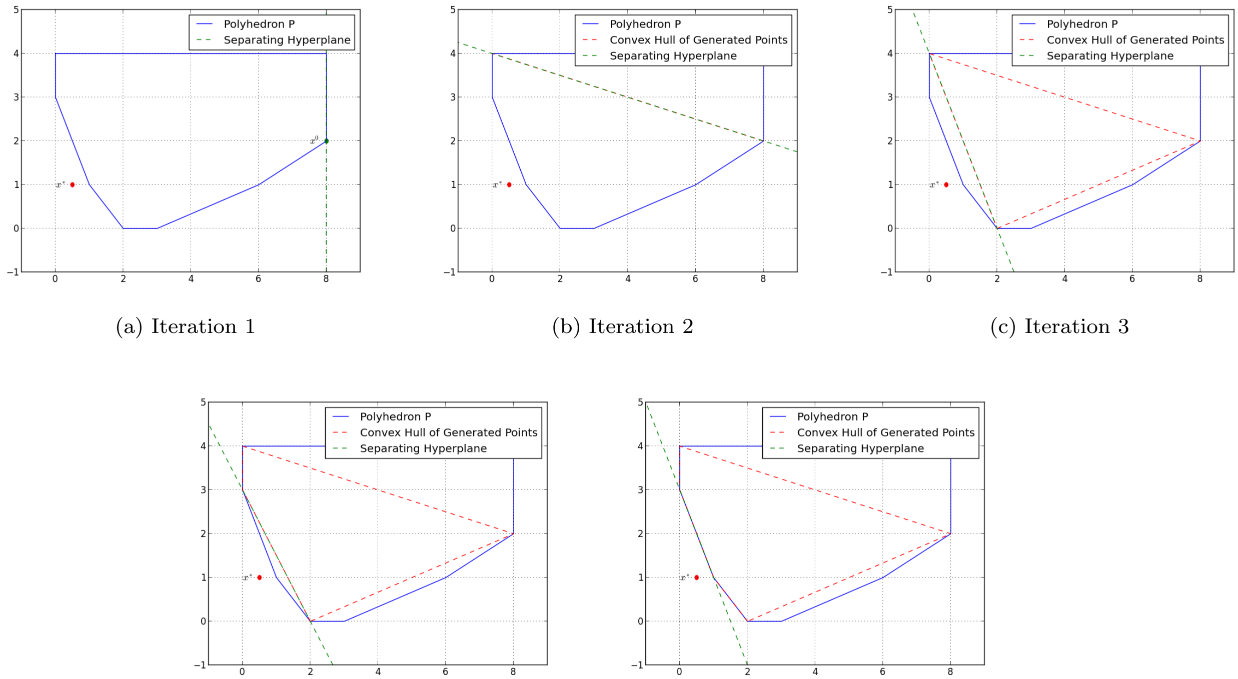
Algorithm 1 Cutting plane for inverse MILP under l_∞ norm

$k \leftarrow 0$, $\mathcal{E}^1 \leftarrow \emptyset$.
do
  $k \leftarrow k + 1$.
  Solve $(InvP_k)$, $d^k \leftarrow d^*$.
  Solve (P_k).
  if (P_k) unbounded **then**
    $y^* \leftarrow \|c\|_\infty$, $d^* \leftarrow 0$, STOP.
  else
    $x^k \leftarrow x^*$.
  end if
  $\mathcal{E}^{k+1} \leftarrow \mathcal{E}^k \cup \{x^k\}$.
while $d^{k\top}(x^0 - x^k) > 0$
$y^* \leftarrow \left\|c - d^k\right\|_\infty$, $d^* \leftarrow d^k$, STOP.



(a) Iteration 1 (b) Iteration 2 (c) Iteration 3

Figure 2: Pictorial illustration of Algorithm 1



(a) Iteration 1 (b) Iteration 2 (c) Iteration 3

(d) Iteration 4 (e) Iteration 5

Figure 3: Pictorial illustration of algorithm for generating Fenchel cut

valid inequalities are sometimes called *Fenchel cuts* [Boyd, 1994]. Figure 3 illustrates how the algorithm for generating Fenchel cuts might proceed for for the same polyhedron as in Figure 2.

A Small Example: Let $c = (-2, 1)$, $x^0 = (0, 3)$ and $\mathcal{S}$ given as in Figure 4 where both x_1 and x_2 are integer and convex hull of $\mathcal{S}$ is given. k, d^k and x^k values through iterations are given in Table 1.

Table 1: k, d^k, x^k and $\mathcal{E}^k$ values through iterations

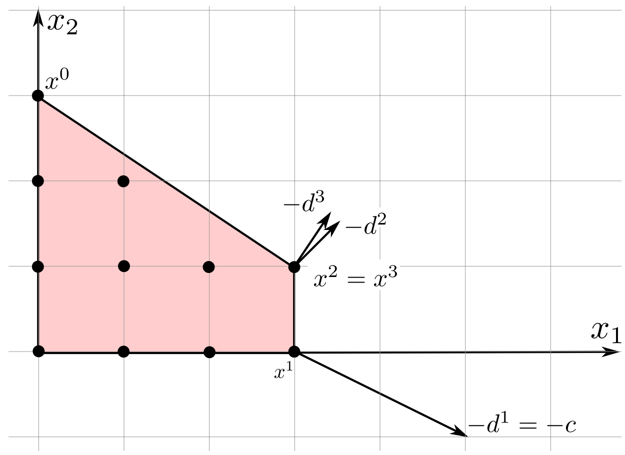
	k	$\mathcal{E}^k$	d^k	x^k	$\|d^k - c\|_\infty$
initialization	1	$\emptyset$	$(-2, 1)$	$(3, 0)$	0
iteration 1	2	$\{(3, 0)\}$	$(-0.5, -0.5)$	$(3, 1)$	1.5
iteration 2	3	$\{(3, 0), (3, 1)\}$	$(-0.4, -0.6)$	$(3, 1)$	1.6

Inverse MILP optimal value is $y^* = \|c - d^3\|_\infty = 1.6$. Inverse MILP optimal solution is $d^3 = (0.4, 0.6)$.

3 Complexity of Inverse MILP

In what follows, we discuss inverse MILP in the traditional framework of computational complexity theory. Before we can describe our results, it will be useful to review some background.

Figure 4: Feasible region and iterations of example problem



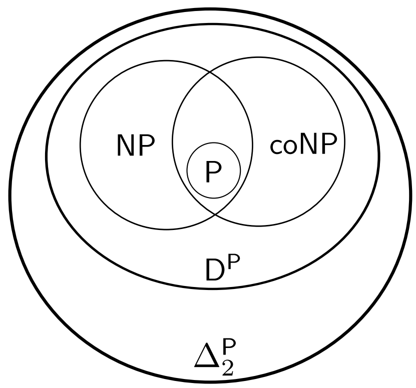


Figure 5: Complexity classes Δ_2^{P}, $\mathsf{D^P}$, NP, coNP and P, assuming $\mathsf{P} \neq \mathsf{NP}$

3.1 Background

The classical NP–completeness theory of Garey and Johnson [1979] addresses decision problems, but for the results we present here, we need to refer to some additional complexity classes. In their contribution to this theory, Papadimitriou and Yannakakis [1982] define the class $\mathsf{D^P}$ to be the class of languages that are the intersection of two languages, the first of which is in NP and the second of which is in coNP. $\mathsf{D^P}$ is a broader class that includes NP and coNP. In their paper, they give examples of problems in $\mathsf{D^P}$ and prove completeness of some of them. One of the problems shown to be complete for $\mathsf{D^P}$ is MILPV, the optimal value problem described below of deciding whether a given value is the exact optimum for a given MILP. We show that the optimal value problem for inverse optimization over is also $\mathsf{D^P}$–complete.

Δ_2^{P} is the class of decision problems that can be solved in polynomial time given an NP oracle. This class is defined as one member of the so-called *polynomial-time hierarchy* in the seminal work of Stockmeyer [1976]. It is a broader class that includes NP, coNP and $\mathsf{D^P}$. Every problem in NP, coNP or $\mathsf{D^P}$ is also in Δ_2^{P}.

Figure 5 illustrates class Δ_2^{P} relative to $\mathsf{D^P}$, NP, coNP and P, assuming $\mathsf{P} \neq \mathsf{NP}$. If $\mathsf{P} = \mathsf{NP}$, we conclude that all classes are equivalent, i.e., $\Delta_2^{\mathsf{P}} = \mathsf{D^P} = \mathsf{NP} = \mathsf{coNP} = \mathsf{P}$. This theoretical possibility is known as the collapse of polynomial hierarchy to its first level [Papadimitriou, 1994], but it is thought to be highly unlikely.

3.2 Complexity of MILP

It will be convenient to refer in what follows to several decision problem associated with forward problem (MILP) and inverse problem (INVMILP). The most commonly associated decision version of forward problem (MILP) is a feasibility problem involving an extra scalar parameter γ, as follows.

Definition 1 ***MILP decision problem (MILPD)****: Given $\gamma \in \mathbb{Q}$, $d \in \mathbb{R}^n$, and an MILP with feasible region $\mathcal{S}$, does there exist $x \in \mathcal{P}$ such that $d^\top x \leq \gamma$?*

It is well-known that this problem is in the complexity class NP–complete and that the optimal solution of problem (MILP) can be determined with a polynomial number of calls to an NP oracle (MILPD) using bissection search. The formal *input* to this decision problem is the quintuplet (A, b, d, r, γ) and the set of such inputs that yields the answer YES is the *language* recognized by an algorithm for solving this problem (formally specified as a Turing machine).

It is useful to recall that a well-known characterization of the class NP is as the class of problems for which the is a *short certificate* for the YES answer. Roughly speaking, a *certificate* is a proof that the answer is correct. A short certificate is one that can be verified in polynomial time. In the case of MILPD, the short certificate is any feasible solution.

The complement of NP is the class coNP of problems for which there is a short certificate for the NO answer. The problem of determining whether a given γ is a lower bound on the value of an optimal solution is an example of a decision problem that is in the class coNP.

Definition 2 ***MILP lower-bounding problem (MILPL)****: Given $\gamma \in \mathbb{Q}$, $d \in \mathbb{R}^n$, and an MILP with feasible region $\mathcal{S}$, is $\min_{x \in \mathcal{P}} d^\top x \geq \gamma$?*

When the answer is NO, a feasible solution in $\mathcal{S}$ with an objective value strictly less than γ is a short certificate. The question of whether a given point is in $\text{conv}(\mathcal{S})$ (membership problem) is equivalent to MILPD. Similarly, asking whether a given inequality is valid (validity problem) for $\text{conv}(\mathcal{S})$ is equivalent to MILPL. The validity problem is itself a membership problem over the 1-polar.

Finally, we consider a third decision problem mentioned earlier, which is that of determining whether the optimal solution value is *exactly* γ.

Definition 3 ***MILP optimal value verification problem (MILPV)****: Given $\gamma \in \mathbb{Q}$, $d \in \mathbb{R}^n$, and an MILP with feasible region $\mathcal{S}$, is $\min_{x \in \mathcal{P}} d^\top x = \gamma$?*

This problem is in the class $\mathsf{D^P}$ of problems defined by Papadimitriou and Yannakakis [1982]. Complexity class $\mathsf{D^P}$ is the class of problems for which the language to be recognized is the intersection of two languages one in class NP and the other in class coNP. Papadimitriou and Yannakakis [1982] showed that MILPV is complete (and (MILP) is hard) for the class $\mathsf{D^P}$.

A related problem is deciding whether a given point is on the boundary of $\text{conv}(\mathcal{S})$. It is also in $\mathsf{D^P}$ since it is equivalent to verifying optimal value.

3.3 Complexity of Inverse MILP

Ahuja and Orlin [2001] show that the inverse problem can be solved in polynomial time when the forward problem is polynomially solvable.

Theorem 1 *(Ahuja and Orlin [2001]) If a forward problem is polynomially solvable for each linear cost function, then the corresponding inverse problems under l_1 and l_∞ norms are polynomially solvable.*

They use the well-known result of Grötschel et al. [1993] to conclude that inverse LP, in particular, is polynomially solvable. Note that this result already indicates that if a given MILP is polynomially solvable, then the associated inverse problem is also polynomially solvable. On the other hand, for general MILPs, the result of Grötschel et al. [1993] cannot tell us precisely what complexity class the inverse problem is in for general MILPs, since the result is about polynomial equivalence, not complexity class equivalence, as we describe in more detail below. The main contribution of this study is to provide the theory that places the inverse problem in the tightest possible class without resolving P versus NP problem. To obtain a formal complexity result, we first consider the decision version of the inverse problem. The decision version is derived in a fashion similar to that of *MILPD*. It asks whether a solution with an objective value less then some given threshold exists.

Definition 4 ***Inverse MILP decision problem (INVD)***: *Given $\gamma \in \mathbb{Q}$, $c \in \mathbb{R}^n$, $x^0 \in \mathbb{R}^n$, and polyhedron $\mathcal{S} \subseteq \mathbb{R}^n$, is the set $\mathcal{K}(\gamma) \cap \mathcal{D}$ non-empty?*

The result of Grötschel et al. [1993] bounds the running time for optimizing a linear objective function over an implicitly defined polyhedron in terms of calls to a separation oracle. Their result can be stated as follows.

Theorem 2 *(Grötschel et al. [1993]) Given an oracle for the separation problem, the optimization problem over a given polyhedron with linear objective can be solved in time polynomial in φ, n and the encoding length of objective coefficient vector, where φ is the facet complexity of the given polyhedron.*

A polyhedron has facet-complexity at most φ if there exists a rational system of inequalities describing the polyhedron in which the encoding length of each inequality is at most φ. The facet complexity thus measures the complexity of a polyhedron independent of its representation. Theorem 2 indicates that, given an oracle for inverse MILP separation, the inverse MILP optimization problem can be solved in time polynomial in φ and n, where the feasible set of (INVMILP-∞) has facet-complexity at most φ, since the objective function of INVMILP-∞ has an encoding length polynomial in n.

To find a bound on φ, consider the third set of constraints of the formulation (INVMILP-∞). The encoding length of the first two sets of constraints depends on the maximum encoding length of c_i, $i \in 1, \dots, n$. The encoding length of the third set of constraints depends on the encoding length of x^0 and the largest encoding length of any extreme point of the convex hull of $\mathcal{S}$. This latter quantity is known as the *vertex complexity* of $\mathcal{S}$ and is a related measure of the complexity of a polyhedron that is bounded by a polynomial function of the facet complexity. Thus, we can say that the running time of the separation–optimization algorithm is polynomial in the encoding length of c_i for $i = 1, \dots, n$, x^0 and the vertex complexity of the convex hull of $\mathcal{S}$. Note that in the case of binary integer optimization problems, the vertex complexity of $\text{conv}(\mathcal{S})$ is always polynomial in n.

These conclusions can be interpreted as stating that the inverse MILP separation problem is equivalent to the MILP optimization problem, but it is important to note that this equivalence is only a polynomial equivalence, not a complexity-wise equivalence. The MILP optimization problem can be solved in polynomial time, given an oracle for the MILP decision problem. Similarly, we conclude that the inverse MILP optimization problem can be solved in polynomial time, given an oracle for the MILP decision problem, which we know to be NP–complete. The following theorem summarizes this result.

Theorem 3 *The inverse MILP optimization problem under the l_∞/l_1 norms is solvable in time polynomial in φ and n, given an oracle for the MILP decision problem.*

This theorem hints at the complexity of inverse optimization problem. We now know that Algorithm 1 solves inverse MILP in polynomial time, given an NP oracle. This algorithm can be used to solve the decision version. In complexity terms. this shows that the inverse problem is in Δ_2^{P}. The following is the restatement of Theorem 3 in complexity terms.

Theorem 4 *INVD under l_1 and l_∞ norms is in Δ_2^{P}.*

The next natural question that comes to mind is whether INVD is complete for this class. Somewhat surprisingly (though not in hindsight), the answer is no. This indicates that GLS result does not yield the tightest complexity class.

The first main result of this paper is the following theorem that shows INVD with an arbitrary norm (l_1, l_∞, or any other p-norm) is in coNP.

Theorem 5 *INVD is in* coNP.

Proof We show existence of a short certificate when the answer to INVD problem is NO. Note that when answer is NO then $\gamma < \|c\|$ (in fact $\gamma < \|c_{\mathcal{S}}\|$, but this is OK for the proof), since $d = 0$ is a valid solution otherwise. Furthermore, when $\gamma = 0$ the problem reduces to MILPL (is $c^\top x^0$ a lower bound for minimization along c over $\mathcal{S}$), which is already known to be in coNP. Therefore it is enough to consider the case where $0 < \gamma < \|c\|$.

When the answer to INVD is NO, then, for each d in $\mathcal{K}(\gamma)$, there exists an x in $\mathcal{S}$ such that $d^\top(x - x^0) < 0$. Hence, the NO answer can be validated by enumeration over $\mathcal{S}$ in principle. What we will show is that we do not need to check the inequality for all x in $\mathcal{S}$, but only for a subset of polynomial size. For this we define the following set first,

$$\mathcal{X}(\gamma) = \{x \in \mathcal{S} \mid \exists d \in \mathcal{K}(\gamma) \text{ s.t. } d^\top(x - x^0) < 0\}.$$

$\mathcal{X}(\gamma)$ is the set of points in $\mathcal{S}$ that are better than x^0 for at least one direction d in $K(\gamma)$. Note that set $\mathcal{X}(\gamma)$ is not empty since answer to problem is NO. Moreover since $\mathcal{X}(\gamma)$ is a subset of $\mathcal{S}$, it is a discrete set. We define another set, $\mathcal{K}^*(\gamma)$, as follows,

$$\mathcal{K}^*(\gamma) = \{x \in \mathbb{R}^n \mid d^\top(x - x^0) \leq 0 \; \forall d \in \mathcal{K}(\gamma)\}.$$

$\mathcal{K}^*(\gamma)$ is the set of points that are better than x^0 for all the directions in $\mathcal{K}(\gamma)$. Note that $\mathcal{K}^*(\gamma)$ is nothing but the dual cone of $\mathcal{K}(\gamma)$ moved along x^0. Both $\mathcal{K}(\gamma)$ and $\mathcal{K}^*(\gamma)$ are full dimensional pointed cones, since $0 < \gamma < \|c\|$.

Cone $\mathcal{K}^*(\gamma)$, set $\mathcal{X}(\gamma)$ and set $\mathcal{S}$ can be considered to be in the primal space, i.e., space of primal solution values. Cones $\mathcal{D}$ and $\mathcal{K}(\gamma)$ can be considered in the dual space, i.e., space of directions.

We claim the following holds when the answer is negative and continue to construct our short certificate. We prove our claim after the short certificate is constructed.

Claim 1 $\operatorname{conv}(\mathcal{X}(\gamma)) \cap \operatorname{int}(\mathcal{K}^*(\gamma)) \neq \emptyset$.

Let $\overline{x} \in \operatorname{conv}(\mathcal{X}(\gamma)) \cap \operatorname{int}(\mathcal{K}^*(\gamma))$, then a subset of $\mathcal{X}(\gamma)$ that can give $\overline{x}$ as a convex combination is a certificate. Moreover, it is a short certificate since we need $n+1$ elements from $\mathcal{X}(\gamma)$ at most. Let $\{x^1, x^2, \ldots, x^k\} \subseteq \mathcal{X}(\gamma)$ be such a subset and $\{\lambda_1, \ldots, \lambda_k\}$ be the corresponding values such that $\overline{x} = \sum_{i=1}^k \lambda_i x^i$, $\sum_{i=1}^k \lambda_i = 1$ and $\lambda_i \geq 0$ for $i = 1, \ldots, k$ and $k \leq n+1$. Next we show how sets $\{x^1, x^2, \ldots, x^k\}$ and $\{\lambda_1, \ldots, \lambda_k\}$ can be used to validate the NO answer.

For any given $d \in \mathcal{D}$, $\overline{x}$ being an element of $\operatorname{int}(K^*(\gamma))$ gives us the following,

$$d^\top \left(\overline{x} - x^0\right) < 0.$$

We can write $\overline{x}$ as a convex combination of x^i values. When we replace $\overline{x}$ using this we get the following inequality,

$$d^\top \left(\sum_{i=1}^k \lambda_i x^i - x^0 \right) < 0.$$

We can manipulate this inequality to get the following inequalities,

$$d^\top \left(\sum_{i=1}^k \lambda_i x^i - \sum_{i=1}^k \lambda_i x^0 \right) < 0,$$
$$\sum_{i=1}^k \lambda_i d^\top \left(x^i - x^0\right) < 0.$$

Then, there exists at least one index j in $\{1, \ldots, k\}$ such that $d^\top(x^j - x^0) < 0$. Then x^j being in $\mathcal{S}$, and being a better solution for direction d means x^0 can not be optimal. Direction d is arbitrary, meaning this result holds for all d in set $\mathcal{K}(\gamma)$. Using sets $\{x^1, \ldots, x^k\}$ and $\{\lambda_1, \ldots, \lambda_k\}$ we validated the NO answer for an arbitrary d in set $\mathcal{K}(\gamma)$. This shows that sets $\{x^1, \ldots, x^k\}$ and $\{\lambda_1, \ldots, \lambda_k\}$ together is a short certificate for the inverse problem defined. $\square$

Proof of Claim 1 Assume $\mathrm{conv}(\mathcal{X}(\gamma)) \cap \mathrm{int}(\mathcal{K}^*(\gamma)) = \emptyset$ for a contradiction. $\mathrm{conv}(\mathcal{X}(\gamma))$ and $\mathcal{K}^*(\gamma)$ are both convex sets. Then there exists a hyperplane that separates these two sets. Let $\{x \in \mathbb{R}^n : a^\top x = \alpha, a \in \mathbb{R}^n, \alpha \in \mathbb{R}\}$ be such a hyperplane that separates $\mathrm{conv}(\mathcal{X}(\gamma))$ and $\mathcal{K}^*(\gamma)$ as follows,

$$a^\top x \geq \alpha \;\forall x \in \mathrm{conv}(\mathcal{X}(\gamma)),$$
$$a^\top x \leq \alpha \;\forall x \in \mathcal{K}^*(\gamma).$$

Then we can write the following inequality,

$$\min_{x \in \mathrm{conv}(\mathcal{X}(\gamma))} a^\top x \geq \max_{x \in K^*(\gamma)} a^\top x. \tag{SEP}$$

Note that problem on the right-hand side is unbounded when a is not in $\mathcal{K}(\gamma)$. Then we can conclude that $a \in \mathcal{K}(\gamma)$ for a valid separating hyperplane. This indicates that x^0 maximizes $a^\top x$ over cone $\mathcal{K}^*(\gamma)$. Then we have the following inequality,

$$\min_{x \in \mathrm{conv}(\mathcal{X}(\gamma))} a^\top x \geq a^\top x^0.$$

Since direction a is in $\mathcal{K}(\gamma)$ and answer to our problem is NO, there exists an $\overline{x}$ in $\mathcal{X}(\gamma)$ such that $a^\top(\overline{x} - x^0) < 0$. Point $\overline{x}$ being a feasible solution for the optimization problem over $\mathrm{conv}(\mathcal{X}(\gamma))$ we have the following inequality,

$$a^\top \overline{x} \geq \min_{x \in \mathrm{conv}(\mathcal{X}(\gamma))} a^\top x \geq a^\top x^0.$$

Using $a^\top(\overline{x} - x^0) < 0$, we can rewrite the inequality as follows,

$$a^\top x^0 > a^\top \overline{x} \geq \min_{x \in \mathrm{conv}(X)(\gamma)} a^\top x \geq a^\top x^0.$$

which is a contradiction. This indicates that the contradiction assumption, existence of a separating hyperplane, is wrong. This proves that $\mathrm{conv}(\mathcal{X}) \cap int(\mathcal{K}^*) \neq \emptyset$. $\square$

Figure 6 shows sets $\mathrm{conv}(\mathcal{S})$, $\mathcal{K}(\gamma)$, $\mathcal{K}^*(\gamma)$ and $\mathrm{conv}(\mathcal{X}(\gamma))$ for the example introduced in Section 2, where c and x^0 are redefined. In this case $c = (-1, -2)$, $x^0 = (2, 1)$ and $\gamma = 1$.

All this theory indicates that complexity of INVD is same as MILPL. The difference is the certificate for MILPL is just a feasible point where the certificate for INVD problem is at most $n + 1$ points with corresponding weights. Certificate for INVD problem is a little more complicated than certificate of MILPL.

Theorem 6 *INVD is* coNP*–complete.*

Proof MILPL can be reduced to INVD. Let inputs of MILPL be $(c, \gamma, \mathcal{S})$ then MILPL can be decided by deciding INVD with inputs $(c^2 \leftarrow c, \gamma^2 \leftarrow 0, \mathcal{S}^2 \leftarrow \mathcal{S}, x^0 \leftarrow \frac{\gamma c}{\|c\|^2})$. INVD asks whether some d in cone $\{d \in \mathbb{R}^n | d^\top \left(\frac{\gamma c}{\|c\|^2} - x\right) \leq 0 \;\forall x \in \mathcal{S}\}$ satisfies $\|c - d\| \leq 0$. Only d that satisfies $\|c - d\| \leq 0$ is $d = c$. For answer to be positive c must be in this cone. c is in this cone if and only if

$$\begin{aligned} c^\top \left(\frac{\gamma c}{\|c\|^2} - x\right) &\leq 0 && \forall x \in \mathcal{S}, \\ \gamma - c^\top x &\leq 0 && \forall x \in \mathcal{S}, \\ \gamma &\leq c^\top x && \forall x \in \mathcal{S}, \end{aligned}$$

which means answer to MILPL is positive. This indicates answer to INVD is positive if and only if answer to MILPL is positive. $\square$

Figure 6: A small example demonstrates conv($\mathcal{S}$), $\mathcal{K}(\gamma)$, $\mathcal{K}^*(\gamma)$, conv($\mathcal{X}(\gamma)$)

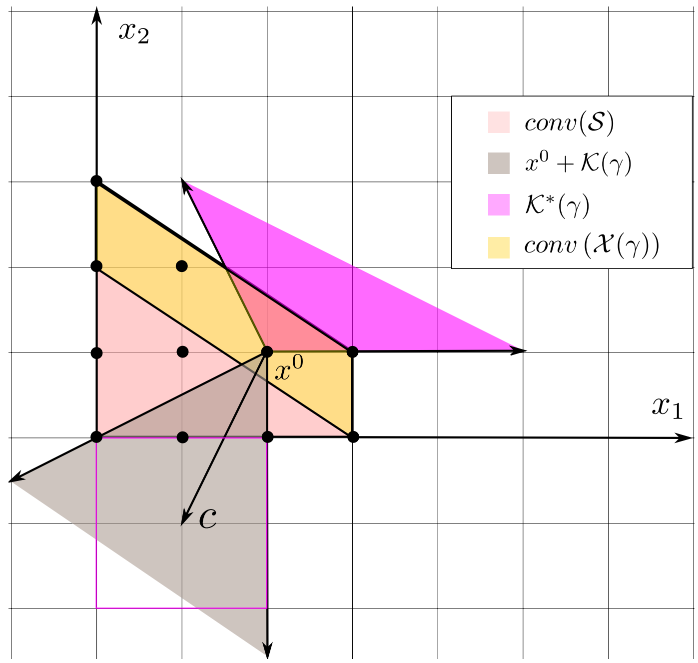


Figure 7: Reduction Example

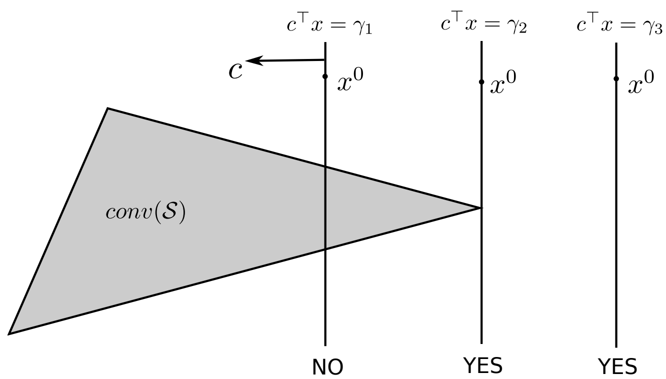


Figure 7 shows x^0 for various γ values. Answer for γ_1 is negative and for γ_2 and γ_3 is positive. Position of x^0 is just for presentation. For γ_1 case x^0 is displayed to be outside of conv($\mathcal{S}$). This is just for display and the result is independent of x^0 being in conv($\mathcal{S}$) or not. The answer is negative for both of the cases.

Lower bound problem for inverse MILP can be defined as follows,

Definition 5 ***Inverse MILP lower-bounding problem (INVL)****: Given $\gamma \in \mathbb{R}$, $c \in \mathbb{R}^n$, $x^0 \in \mathbb{R}^n$ and an MILP with feasible set $\mathcal{S}$, is $\min_{d \in \mathcal{K}(y) \cap \mathcal{D}} y \geq \gamma$?*

Theorem 7 *INVL problem is in* NP.

Proof We need to show existence of a short certificate that can validate YES answer. When answer is YES, optimal value of the inverse problem is greater than equal to γ. We show existence of a short certificate that validates optimal value can not be less than γ, i.e. no feasible direction d that optimizes x^0 and its distance to c is less than γ. This is same as validating NO answer for INVD. The only difference is now the directions are strictly less than γ-distance to c. Remember the claim we proved,

$$\text{conv}(\mathcal{X}(\gamma)) \cap int(\mathcal{K}^*(\gamma)) \neq \emptyset.$$

Note that $\mathcal{K}^*(\gamma)$ is the set of points that are at least as good as x^0 for all directions d at most γ-distant to c. $int(\mathcal{K}^*(\gamma))$ is the set of points that are strictly better than x^0 for all directions d that are strictly less than γ-distant to c.

After this point the proof goes on same as proof of INVD being in coNP. The short certificate is the same and it can be used to show that the optimal value of inverse problem can not be less than γ.

Theorem 8 *INVL is* NP*–complete.*

Claim 2 *There exists a positive ϵ, such that if $c^\top x > \gamma$ holds for all x in $\mathcal{S}$ then $c^\top x > \gamma + \epsilon$ holds for all x in $\mathcal{S}$.*

Proof of Claim 2 Such an ϵ can be found using vertex complexity of $\mathcal{S}$. Its encoding will be a polynomial of c and vertex complexity of $\mathcal{S}$.

Claim 3 *If there exists an x in $\mathcal{S}$ such that $\gamma \geq c^\top x$ holds, then one can find a positive δ such that $\|c - d\| \geq \delta$ holds for all feasible d for the inverse problem with the following input ($c^2 \leftarrow c$, $\mathcal{S}^2 \leftarrow \mathcal{S}$, $x^0 \leftarrow \frac{(\gamma+\epsilon)c}{\|c\|^2}$).*

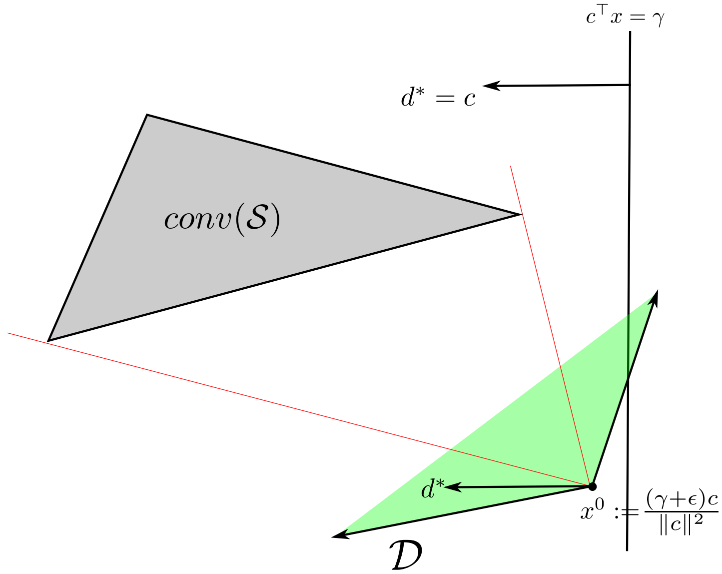
Proof of Claim 3 To prove the claim we will manipulate the inverse problem constraint. Inverse problem constraint is as follows,

$$d^\top(x^0 - x) \leq 0 \;\forall x \in \mathcal{S}.$$

Let c optimizes $\overline{x}$ over $\mathcal{S}$. Inverse problem constraint will hold for $\overline{x}$. We can write it using $\overline{x}$ as follows,

$$\begin{aligned}
d^\top(x^0 - \overline{x}) &\leq 0 \\
d^\top(x^0 - \overline{x}) - c^\top(x^0 - \overline{x}) + c^\top(x^0 - \overline{x}) &\leq 0 \\
(d - c)^\top(x^0 - \overline{x}) &\leq -c^\top x^0 + c^\top \overline{x} \\
(d - c)^\top(x^0 - \overline{x}) &\leq -\gamma - \epsilon + c^\top \overline{x} \\
\gamma + \epsilon - c^\top \overline{x} &\leq (c - d)^\top(x^0 - \overline{x}) \\
\epsilon &\leq (c - d)^\top(x^0 - \overline{x}) \\
\epsilon &\leq \|c - d\|\|x^0 - \overline{x}\| \\
\frac{\epsilon}{\|x^0 - \overline{x}\|} &\leq \|c - d\|.
\end{aligned}$$

Figure 8: Claim 2 on a Simple Example



Using γ, c and vertex complexity of $\mathcal{S}$, such a positive δ can be computed.

Proof of Theorem 8 MILPD can be reduced to INVL. Let inputs of MILPD be $(c,\, \gamma,\, \mathcal{S})$ then MILPD can be resolved by deciding INVL with inputs $(c^2 \leftarrow c,\, \gamma^2 \leftarrow \delta,\, \mathcal{S}^2 \leftarrow \mathcal{S},\, x^0 \leftarrow \frac{(\gamma+\epsilon)c}{\|c\|^2})$. ϵ and δ are small positive rationals computed from inputs of MILPD as explained in Claim 2 and Claim 3.

INVL asks whether $\|c - d\| \geq \delta$ holds for all d in $\{d \in \mathbb{R}^n | d^\top \left(\frac{(\gamma+\epsilon)c}{\|c\|^2} - x\right) \leq 0 \;\forall x \in \mathcal{S}\}$.

Deciding INVL with described inputs resolves MILPD. If answer to INVL is positive, then $d = c$ is not feasible for inverse problem. This indicates that c does not optimize x^0 over $\mathcal{S}$, there exists $\overline{x}$ in $\mathcal{S}$ such that,

$$c^\top \overline{x} < c^\top x^0 = \gamma + \epsilon.$$

Using Claim 2 we can deduct $c^\top \overline{x} \leq \gamma$. This means answer to MILPD is positive.

When answer to INVL is negative, optimal value of inverse problem is 0 by our design of δ, Claim 3. This indicates c optimizes x^0,

$$\gamma < c^\top x^0 = \gamma + \epsilon < c^\top x \quad \forall x \in \mathcal{S}.$$

This means answer to MILPD is negative. □

Note that the reduction presented in Theorem 8 can also be used in Theorem 6. The one presented in Theorem 6 is just simpler and does not require introduction of ϵ and δ.

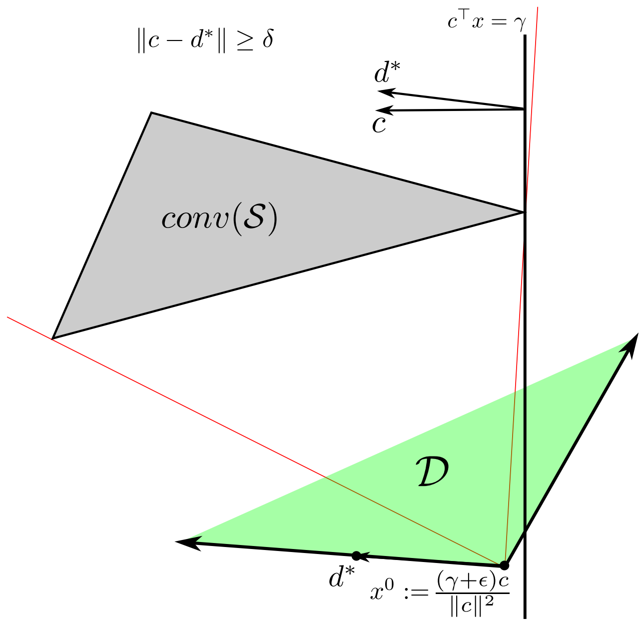
Figure 8 illustrates the case described in Claim 2, inequality $c^\top x > \gamma$ holds for all x in $\mathcal{S}$. Figure displays the cone of feasible d directions and optimal d as d^*. Answer to both MILPD and INVL problems is negative.

Figure 9 illustrates a case where optimal value of MILP is exactly γ. It can also be considered as an illustration of the case described in Claim 3, forward problem optimal value is exactly gamma. Inverse optimal value is denoted by d^*. Positive δ as described in Claim 3 is a lower bound for the inverse problem. Answer to both MILPD and INVL problems for the displayed inputs is positive. From the figure it is easy to see that result in Claim 3 holds when the forward problem optimal value is not exactly γ but strictly less.

Definition 6 ***Inverse MILP optimal value verification problem (INVO)****: Given $\gamma \in \mathbb{Q}$, $x^0 \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, and an MILP with feasible region $\mathcal{S}$, is $\min_{d \in \mathcal{K}(y) \cap \mathcal{D}} y = \gamma$?*

Theorem 9 *INVO problem is in class D^P–complete.*

Figure 9: Claim 3 on a Simple Example



Proof As noted before, reduction presented in Theorem 8 can be used to reduce both MILPL and MILPD to INVD and INVL problems respectively. Using this reduction, language of INVO can be written as an intersection of languages of INVD and INVL that are in coNP and NP respectively. This proves that INVO is in class $\mathsf{D^P}$. INVO problem is complete for $\mathsf{D^P}$ since MILPV can be reduced to INVO using the same reduction. □

Note that verifying exact optimal value of both inverse and forward problems are $\mathsf{D^P}$–complete.

4 Conclusion and Future Directions

In this paper, we formally defined various problems related to the inverse MILP problem in which we try to derive an objective function d closest to a given estimate c that make a given solution x^0 optimal over the feasible region $\mathcal{S}$ to an MILP. This problem can be seen as an optimization problem over the set of all inequalities valid for $\mathcal{S}$ and satisfied at equality by x^0. Alternatively, it can also be seen as optimization over the 1-polar with some additional constraints. Both these characterization make the connection the separation problem associated with $\mathcal{S}$ evident.

After defining the problem formally, we gave a cutting plane algorithm for solving it under the l_1 and l_∞ norms and observed that the separation problem for the feasible region is equivalent to the original forward problem, enabling us to conclude by the framework of Grötschel et al. [1993] that the problem can be solved with a polynomial number of calls to an oracle for solving the forward problem.

This algorithm places the decision version of inverse MILP in the complexity class Δ_2^{P}, but it is possible to prove a stronger result. The main contribution of this study is to show that this decision problem is complete for the class coNP, which is on the same level of the polynomial-time hierarchy of that of the forward problem. We proved the problem is in coNP by giving a short certificate for the negative answer and then show it is complete for coNP by reducing the MILP lower bound (MILPL) problem to inverse MILP decision (INVD) problem. We also provide a reduction for the inverse lower bound problem. Finally, we show that the inverse optimal value verification problem is complete to the class $\mathsf{D^P}$, which is precisely the same class containing the MILP optimal value verification problem.

Theorem 2 states that an optimization problem (over a convex set) can be solved in polynomial time given an oracle for the separation problem. Technically, this does not allow us to place the optimization and separation problems on precisely the same level of the polynomial hierarchy. It is likely that the GLS result can be modified slightly in order to show that optimization and separation are indeed on the same level of the hierarchy. There are also some interesting open questions remaining to be explored with respect to complexity.

Finally, we have implemented the algorithm and a computationally oriented study is left as future work. Such a study would reveal the practical performance of the separation-optimization procedure and investigate the possible relationship between the number of iterations (oracle calls) and the polyhedral complexity (vertex/facet complexity), among other things. This may provide practical estimates for the number of iterations required to solve certain classes of problems.

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