Tools for Modeling Optimization Problems
A Short Course

Modeling Concepts

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What will this workshop be about?

- **Modeling** basic financial optimization problems as mathematical optimization problems.
- Expressing those optimization problems in an algebraic modeling language.
- Getting data into the model, especially through the use of spreadsheets.
- The use of Python as both a modeling tool and as a wrapper for creating applications that populate models with data, perform post-solution analysis and even chain multiple models together.
- **Advanced techniques** for development of more sophisticated models and methods of analysis. techniques.
What will this workshop NOT be about?

• I’ll assume you know the basic principles behind the models we’ll look at.

• I will not discuss (much of) the theory behind the models.

• The models I’ll show are relatively unsophisticated—their purpose is only to illustrate basic principles of algebraic modeling.

• I won’t be able to cover Python as a programming language in much detail, but I hope to leave you with enough knowledge to get started.
The (Open Source) Toolbox

- COIN-OR
- SolverStudio (Excel plug-in)
- Portable Python
- Eclipse
- PyDev (Eclipse plug-in)
- (AMPL)
What is Mathematical Optimization?

• A *mathematical optimization problem* consists of
  – a set of *variables* that describe the state of the system,
  – a set of *constraints* that determine the states that are allowable,
  – external input *parameters* and *data*, and
  – an *objective function* that provides an assessment of how well the system is functioning.

• The variables represent operating *decisions* that must be made.

• The constraints represent operating *specifications*.

• The goal is to determine the *best* operating state consistent with specifications.
Forming a Mathematical Programming Model

The general form of a mathematical programming model is:

\[ \min \text{ or } \max f(x_1, \ldots, x_n) \]
\[ \text{s.t. } \begin{array}{c}
g_i(x_1, \ldots, x_n) \leq b_i, \ i \in M \\
g_i(x_1, \ldots, x_n) = b_i, \ i \in M \\
g_i(x_1, \ldots, x_n) \geq b_i, \ i \in M \\
\end{array} \]
\[ (x_1, \ldots, x_n) \in X \]

\( X \) may be a discrete set, such as \( \mathbb{Z}^n \).

Notes:

- There is an important assumption here that all input data are known and fixed.
- Such a mathematical program is called deterministic.
- Is this realistic?
Categorizing Mathematical Optimization Problems

- Mathematical optimization problems can be categorized along several fundamental lines.
  - **Constrained** vs. **Unconstrained**
  - **Convex** vs. **Nonconvex**
  - **Linear** vs. **Nonlinear**
  - **Discrete** vs. **Continuous**

- What is the importance of these categorizations?
  - Knowing what category an instance is in can tell us something about how difficult it will be to solve.
  - Different solvers are designed for different categories.
Unconstrained Optimization

• When \( M = \emptyset \) and \( X = \mathbb{R}^n \), we have an unconstrained optimization problem.

• Unconstrained optimization problems will not generally arise directly from applications.

• They do, however, arise as subproblems when solving constrained optimization problems.

• In unconstrained optimization, it is important to distinguish between the convex and nonconvex cases.

• In the convex case, optimizing globally is “easy.”
Definition 1. Let $S$ be a nonempty convex set on $\mathbb{R}^n$. Then the function $f : S \rightarrow \mathbb{R}$ is said to be convex on $S$ if

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

for each $x_1, x_2 \in S$ and $\lambda \in (0, 1)$.

- **Strictly convex** means the inequality is strict.
- **(Strictly) concave** is defined analogously.
- A function that is neither convex nor concave is called **nonconvex**.
- A **linear** function is one that is both concave and convex.
Example: Convex and Nonconvex Functions

CONVEX

NONCONVEX
(Constrained) Linear Optimization

• A *linear program* is one that can be written in a form in which the functions \( f \) and \( g_i, i \in M \) are all linear and \( X = \mathbb{R}^n \).

• In general, a linear program is one that can be written as

\[
\begin{align*}
\text{minimize} & \quad c^\top x \\
\text{s.t.} & \quad a_i^\top x \geq b_i \ \forall i \in M_1 \\
& \quad a_i^\top x \leq b_i \ \forall i \in M_2 \\
& \quad a_i^\top x = b_i \ \forall i \in M_3 \\
& \quad x_j \geq 0 \ \forall j \in N_1 \\
& \quad x_j \leq 0 \ \forall j \in N_2 
\end{align*}
\]

• Equivalently, a linear program can be written as

\[
\begin{align*}
\text{minimize} & \quad c^\top x \\
\text{s.t.} & \quad Ax \geq b
\end{align*}
\]

• Generally speaking, linear optimization problems are also “easy” to solve.
(Constrained) Nonlinear Optimization

- A *nonlinear program* is any mathematical program that cannot be expressed as a linear program.
- Usually, this terminology also assumes $X = \mathbb{R}^n$.
- Note that by this definition, it is not always obvious whether a given instance is really nonlinear.
- In general, nonlinear optimization problems are difficult to solve to global optimality.
Special Case: Convex Optimization

- A **convex optimization problem** is a nonlinear optimization problem in which the objective function $f$ is convex and the feasible region

$$\mathcal{F} = \{x \in \mathbb{R}^n \mid g_i(x) \geq b_i\}$$

is a convex set.

- In practice, convex programs are usually “easy” to solve.
Convex Sets

A set $S$ is **convex**

\[ x_1, x_2 \in S, \lambda \in [0, 1] \Rightarrow \lambda x_1 + (1 - \lambda) x_2 \in S \]

- If $y = \sum \lambda_i x_i$, where $\lambda_i \geq 0$ and $\sum \lambda_i = 1$, then $y$ is a **convex combination** of the $x_i$’s.

- If the positivity restriction on $\lambda$ is removed, then $y$ is an **affine combination** of the $x_i$’s.

- If we further remove the restriction that $\sum \lambda_i = 1$, then we have a **linear combination**.
Example: Convex and Nonconvex Sets
Special Case: Quadratic Optimization

• If all of the functions $f$ and $g_i$ for $i \in M$ are quadratic functions, then we have a quadratic program.

• Often, the term quadratic program refers specifically to a program of the form

$$\text{minimize} \quad \frac{1}{2} x^\top Q x + c^\top x$$

s.t. \quad Ax \geq b$

• Because $x^\top Q x = \frac{1}{2} x^\top (Q + Q^\top) x$, we can assume without loss of generality that $Q$ is symmetric.

• The objective function of the above program is then convex if and only if $Q$ is positive semidefinite, i.e., $y^\top Q y \geq 0$ for all $y \in \mathbb{R}^n$.

• There are specialized methods for solving convex quadratic optimization problems efficiently.
Special Case: Discrete Optimization

- When $X = \mathbb{Z}^n$, we have a discrete or integer optimization problem.
- When $X = \mathbb{Z}^r \times \mathbb{R}^{n-r}$, we have a mixed integer optimization problem.
- If some of the functions are nonlinear, then we have a mixed integer nonlinear optimization problem.
- Almost all mathematical optimization with integer variables are difficult to solve.
Local versus Global Optimization

• For a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, a *local minimizer* is an $\hat{x} \in \mathbb{R}^n$ such that $f(\hat{x}) \leq f(x)$ for all $x$ in a *neighborhood* of $\hat{x}$.

• In general, it is “easy” to find local minimizers.

• A *global minimizer* is $x^* \in \mathbb{R}^n$ such that $f(x^*) \leq f(x)$ for all $x \in \mathbb{R}$.

• The importance of convexity is the following:

**Theorem 1.** For a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, if $x^* \in \mathbb{R}^n$ is a local optimal solution to $\min_{x \in \mathbb{R}^n} f(x)$, then $x^* \in \mathbb{R}^n$ is also a global optimal solution.
Solutions

• What is the result of analyzing a mathematical program?

• A solution is an assignment of values to variables.

• A solution can be thought of as a vector.

• A feasible solution is an assignment of values to variables such that all the constraints are satisfied.

• The objective function value of a solution is obtained by evaluating the objective function at the given solution.

• An optimal solution (assuming minimization) is one whose objective function value is less than or equal to that of all other feasible solutions.

• We may also be interested in some additional qualities of the solution.
  – Sensitivity
  – Robustness
  – Risk
Stochastic Optimization

• In the real world, little is known ahead of time with certainty.

• A \textit{risky investment} is one whose return is not known ahead of time.

• a \textit{risk-free} investment is one whose return is fixed.

• To make decision involving risky investments, we need to incorporate some degree of \textit{stochasticity} into our models.
Types of Uncertainty

• Where does uncertainty come from?
  – Weather
  – Financial Uncertainty
  – Market Uncertainty
  – Competition
  – Technology

• In decision analysis, we must proceed through this list and identify items that might affect the outcome of a decision.
The Scenario Approach to Uncertainty

• The *scenario approach* assumes that there are a finite number of possible future outcomes of uncertainty.

• Each of these possible outcomes is called a *scenario*.
  
  – Demand for a product is “low, medium, or high.”
  – Weather is “dry or wet.”
  – The market will go “up or down.”

• Even if this is not reality, often a discrete approximation is useful.
Multi-period Optimization Models

• When we introduce time as an element of a stochastic optimization model, we also have to address the following questions.
  – When do we have to make a given decision?
  – What will we know at the time we are making the decision?
  – How far into the future are we looking?

• Multi-stage models generally assume that decision are made in stages and that in each stage, some amount of uncertainty is resolved.

• Example
  – Fred decides to rebalance his investment portfolio once a quarter.
  – At the outset, he only knows current prices and perhaps some predictions about future prices.
  – At the beginning of each quarter, prices have been revealed and Fred gets a chance to make a “recourse decision.”
Example 1: Short Term Financing

A company needs to make provisions for the following cash flows over the coming five months: $-150K$, $-100K$, $200K$, $-200K$, $300K$.

- The following options for obtaining/using funds are available,
  - The company can borrow up to $100K$ at 1% interest per month,
  - The company can issue a 2-month zero-coupon bond yielding 2% interest over the two months,
  - Excess funds can be invested at 0.3% monthly interest.

- How should the company finance these cash flows if no payment obligations are to remain at the end of the period?
Example 1 (cont.)

• All investments are risk-free, so there is no stochasticity.

• What are the decision variables?
  – $x_i$, the amount drawn from the line of credit in month $i$,
  – $y_i$, the number of bonds issued in month $i$,
  – $z_i$, the amount invested in month $i$,

• What is the goal?
  – To maximize the the cash on hand at the end of the horizon.
Example 1 (cont.)

The problem can then be modelled as the following linear program:

\[
\begin{align*}
\max_{(x, y, z, v) \in \mathbb{R}^{12}} & \quad f(x, y, z, v) = v \\
\text{s.t.} & \quad x_1 + y_1 - z_1 = 150 \\
& \quad x_2 - 1.01x_1 + y_2 - z_2 + 1.003z_1 = 100 \\
& \quad x_3 - 1.01x_2 + y_3 - 1.02y_1 - z_3 + 1.003z_2 = -200 \\
& \quad x_4 - 1.01x_3 - 1.02y_2 - z_4 + 1.003z_3 = 200 \\
& \quad -1.01x_4 - 1.02y_3 - v + 1.003z_4 = -300 \\
& \quad 100 - x_i \geq 0 \quad (i = 1, \ldots, 4) \\
& \quad x_i \geq 0 \quad (i = 1, \ldots, 4) \\
& \quad y_i \geq 0 \quad (i = 1, \ldots, 3) \\
& \quad z_i \geq 0 \quad (i = 1, \ldots, 4) \\
& \quad v \geq 0.
\end{align*}
\]
Example 2: Portfolio Optimization

- We have the choice to invest in either Intel, Wal-Mart, or G.E..
- All stocks currently cost $100/share.
- In one year’s time, the market will be either “up” or “down.”
- the Following are the possible outcomes

<table>
<thead>
<tr>
<th></th>
<th>Intel</th>
<th>Pepsico</th>
<th>Wal-Mart</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>130</td>
<td>112</td>
<td>115</td>
</tr>
<tr>
<td>Down</td>
<td>90</td>
<td>108</td>
<td>103</td>
</tr>
</tbody>
</table>

- What would you do?
Example 2 (cont.)

- If the probability that the market is up is $p$, then the value at year 1 from investing $1000 are as follows:
  - Intel: $900 + 400p$
  - Pepsico: $1080 + 40p$
  - Wal-Mart: $1030 + 120p$

- What is your objective?