Computational Integer Programming
Universidad de los Andes

Lecture 5

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Reading for This Lecture

- Wolsey, Chapters 10 and 11
- Nemhauser and Wolsey Sections II.3.1, II.3.6, II.3.7, II.5.4
- "Decomposition in Integer Programming," Ralphs and Galati.

The Decomposition Principle

- Again, we consider a pure integer program IP defined by

$$\begin{aligned} z_{IP} &= \max\{cx \mid x \in S\}, \\ S &= \{x \in \mathbb{Z}^n_+ \mid Ax \leq b\}. \end{aligned}$$

- We also assume all variables have finite upper and lower bounds.
- Recall the concept of Lagrangian relaxation: we relax some constraints and then penalize their violation.
- The *principle of decomposition* is to divide the inequalities describing S into two sets:
  - the "easy constraints," and
  - the "complicating constraints," and

  is such a way that removing the complicating constraints results in a integer program we can solve effectively.

The Lagrangian Relaxation

- Suppose as before that our IP is defined by

$$\begin{aligned} \max\ & cx \\ \text{s.t. } A^1x &\leq b^1 \text{ (the ``complicating'' constraints)} \\ A^2x &\leq b^2 \text{ (the ``nice'' constraints)} \\ x &\in \mathbb{Z}^n \end{aligned}$$

  where optimizing over $S_{LR} = \{x \in \mathbb{Z}^n \mid A^2x \leq b^2\}$ is "easy."

- Lagrangian Relaxation (for $u \geq 0$):

$$LR(u) : z_{LR}(u) = ub^1 + \max_{x \in S_{LR}} \{(c - uA^1)x\}.$$

The Lagrangian Dual

- The next step is to obtain a dual problem formed by allowing u to vary.
- We are looking for the value of $u \geq 0$ that yield the lowest upper bound.
- The Lagrangian dual problem, LD, is

$$z_{LD} = \min_{u \geq 0} z_{LR}(u)$$

- The Lagrangian dual can be rewritten as the following LP

$$z_{LD} = \min_{\eta, u} \{\eta + ub^1 \mid \eta \geq (c - uA^1)x^i, i \in 1, \ldots, T, u \geq 0\}$$

  where $\{x^i\}_{i=1}^T$ are the extreme points of $\text{conv}(S_{LR})$.

- This can be solved using a cutting plane algorithm where the separation problem is an optimization problem over the set S_{LR}.

Solving the Lagrangian Dual with Subgradient Optimization

- Note that $(c - uA^1)x$ is an affine function of u for a fixed x.
- This tells us that $z_{LR}(u)$, when viewed as a function of u, is the maximum of a finite number of affine functions.
- Hence, it is piecewise linear and convex on the domain over which it is finite.
- We can easily minimize any convex function which we can evaluate and subdifferentiate using a technique called *subgradient optimization*.
- This technique is covered in detail in nonlinear programming.
- The procedure iteratively adjusts the weights according to the degree of violation of each constraint.

Subgradient Algorithm for the Lagrangian Dual

- The idea of the subgradient algorithm is to first fix u and determine x by optimizing over S_{LR}.
- Then update u according to the observed violations.
- Here is a basic *subgradient algorithm* for solving the Lagrangian dual:
  1. Choose initial Lagrange multipliers $u^0 \geq 0$ and set $t = 0$.
  2. Solve the Lagrangian subproblem $LR(u^t)$.
  3. Calculate the current violation of the complicating constraints $s = b^1 - A^1 x$.
  4. Set $u_j^{t+1} \leftarrow \max\{u_j^t - \mu^t \frac{s_j}{\|s\|}, 0\}$ where μ^t is the chosen *step size*.
  5. Set $t \leftarrow t + 1$ and go to step 2.
- This algorithm is guaranteed to converge to the optimal solution as long as $\{\mu^t\}_{t=0}^{\infty} \rightarrow 0$ and $\sum_{t=0}^{\infty} \mu^t = \infty$
- In practice, one usually uses a geometric progression for the step sizes.
- Sometimes, it's difficult to know when the optimal solution has been reached.

Dantzig-Wolfe Decomposition

- In this technique, we utilize the fact that every point in $\text{conv}(S_{LR})$ can be written as the convex combination of extreme points of $\text{conv}(S_{LR})$.
- Here is the Dantzig-Wolfe LP:

$$\begin{aligned} \max \quad & \sum_{i=1}^{T} cx^i \lambda^i \\ s.t. \quad & \sum_{i=1}^{T} A^1 x^i \lambda^i \leq b^1 \\ & \sum_{I=1}^{T} \lambda^i = 1 \\ & \lambda \in \mathbb{R}_+^T \end{aligned}$$

where $\{x^i\}_{i=1}^T$ are the extreme points of $\text{conv}(S_{LR})$.

- This is a relaxation of IP; solving yields an upper bound.

Solving the Dantzig-Wolfe LP

- We can solve this LP using column generation.
- The column generation subproblem is again an optimization problem over S_{LR}.
- Note that this LP is exactly the dual of the LP we derived as being equivalent to the Lagrangian dual!
- Hence, this gives the same bound as the Lagrangian dual.

Comparing Dantzig-Wolfe to Lagrangian Relaxation

- Because they are conceptually equivalent, the distinction between Dantzig-Wolfe and Lagrangian relaxation is a bit artificial.
- Philosophically, the distinction between them is in the solution methodology typically applied and in the form of the output.
- The Lagrangian dual produces only a dual solution and does not include any explicit primal solution information.
- Dantzig–Wolfe is required to produce both a primal and a dual solution.
- The primal solution information can be used to perform separation and tighten the relaxation.

The Strength of the Decomposition Bound

- We can characterize its strength of the bound obtained by decomposition as follows:

$$z_D = \max\{cx \mid A^1 x \leq b^1, x \in \text{conv}(S_{LR})\}$$

- Using this fact, we can characterize exactly when the decomposition bound is strong.

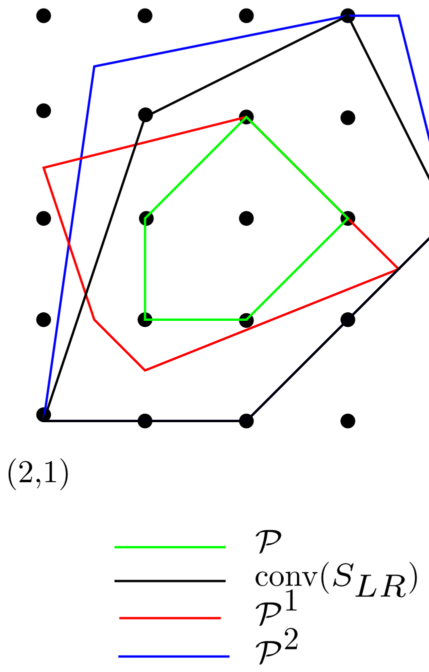
Proposition 1. $z_{IP} = z_D$ *for all objective functions if and only if*

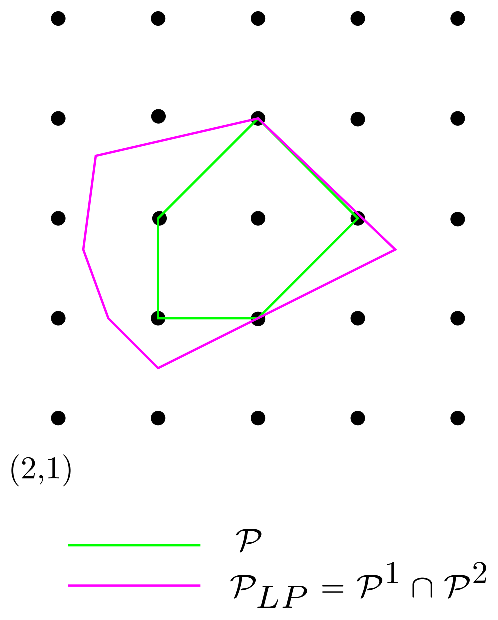
$$\textit{conv}\{S_{LR} \cap \{x \in \mathbb{R}^n_+ \mid A^1 x \leq b^1\}\} = \textit{conv}(S_{LR}) \cap \{x \in \mathbb{R}^n_+ \mid A^1 x \leq b^1\}$$

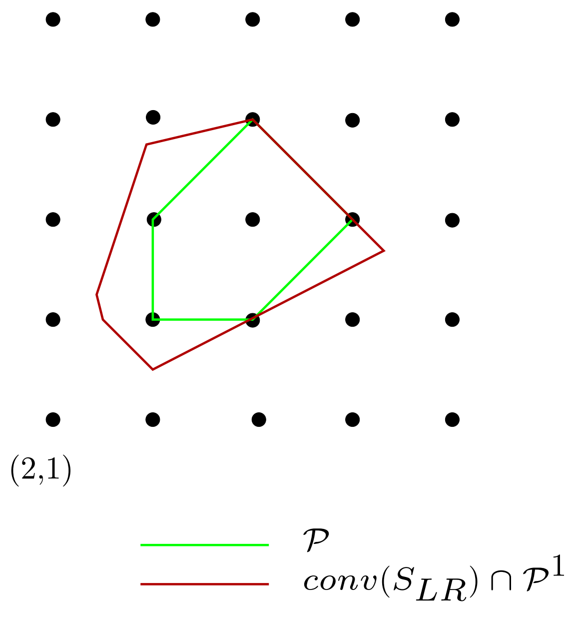
Example

$$
\begin{aligned}
\min \quad & x_1 \\
-x_1 - x_2 &\geq -8, && (1) \\
-0.4x_1 + x_2 &\geq 0.3, && (2) \\
x_1 + x_2 &\geq 4.5, && (3) \\
3x_1 + x_2 &\geq 9.5, && (4) \\
0.25x_1 - x_2 &\geq -3, && (5) \\
7x_1 - x_2 &\geq 13, && (6) \\
x_2 &\geq 1, && (7) \\
-x_1 + x_2 &\geq -3, && (8) \\
-4x_1 - x_2 &\geq -27, && (9) \\
-x_2 &\geq -5, && (10) \\
0.2x_1 - x_2 &\geq -4, && (11) \\
x &\in \mathbb{Z}^2. && (12)
\end{aligned}
$$

Illustrating the Strength of the Lagrangian Dual







$$\begin{aligned}
\mathcal{P} &= \text{conv}\{x \in \mathbb{Z}^2 \mid x \text{ satisfies } (1) - (11)\}, \\
\mathcal{P}^1 &= \{x \in \mathbb{R}^2 \mid x \text{ satisfies } (1) - (5)\}, \text{ and} \\
\mathcal{P}^2 &= \{x \in \mathbb{R}^2 \mid x \text{ satisfies } (6) - (11)\}, \\
S_{LR} &= \mathcal{P}^2 \cap \mathbb{Z}^2.
\end{aligned}$$

Comparing the Decomposition Bound to the LP bound

- The following proposition follows again from the characterization of z_{LD}.

  Proposition 2. *The LP relaxation of IP gives the bound z_D for all objective functions if $\{x \in \mathbb{R}^n_+ \mid A^2 x \leq b^2\}$ is an integral polyhedron.*

- This follows from the fact that $\text{conv}(S_{LR}) = \{x \in \mathbb{R}^n_+ \mid A^2 x \leq b^2\}$ in this case.
- Because of the equivalence of optimization and separation, we can in theory always attain this bound using a cutting plane algorithm (why?).
- However, in some cases, decomposition methods can compute this bound more efficiently.
- The advantage of the LP relaxation is that it can be further strengthened using cutting planes valid for S.
- It is also possible to strengthen the Lagrangian dual in this way.

Choosing a Decomposition

- Often, there are multiple choices for the decomposition.
- The definition of the set S_{LR} determines the strength of the bound.
- However, it is important to choose a relaxation that can be solved relatively easily (but not too easily).
- The relaxation must be solved iteratively in order to obtain the bound.
- Recall the TSP example.

Comparing Decomposition-based Bounding to LP-based Bounding

- The class of methods we have just discussed are called *decomposition-based methods* because they decompose the problem into two parts.
- Up until the mid-1970's, these methods were very popular for solving integer programming problems.
- They can effectively strengthen the bound obtained by LP relaxation alone.
- However, after methods based on strengthening the LP relaxation using polyhedral cutting planes were introduced, these methods fell out of favor.
- It is possible to combine these two approaches.
- This is one of the current frontiers of research in integer programming.