References for This Lecture

- Wolsey Chapter 2
- Nemhauser and Wolsey Sections II.3.1, II.3.6, II.4.1, II.4.2, II.5.4
Relaxation

For simplicity, we now consider a pure integer program $IP$ defined by

$$z_{IP} = \max\{cx \mid x \in S\},$$

$$S = \{x \in \mathbb{Z}^n_+ \mid Ax \leq b\}.$$ 

Definition 1. A relaxation of $IP$ is a maximization problem defined as

$$z_R = \max\{z_R(x) \mid x \in S_R\}$$

with the following two properties:

$$S \subseteq S_R \quad \text{(1)}$$

$$cx \leq z_R(x), \forall x \in S. \quad \text{(2)}$$
Importance of Relaxations

- The main purpose of a relaxation is to obtain an upper bound on $z_{IP}$.
- Relaxation is used as a method of bounding in branch and bound.
- The idea is to choose a relaxation that is much easier to solve than the original problem.
- Note that the relaxation must be solved to optimality to yield a valid bound.
- We first consider three basic types of relaxations.
  - LP relaxation
  - Combinatorial relaxation
  - Lagrangian relaxation
- Relaxations are also used in some other bounding schemes we will look at.
Aside: How Do You Spell “Lagrangian?”

- Some spell it “Lagrangean.”
- Some spell it “Lagrangian.”
- We ask Google.

- In 2002:
  - “Lagrangean” returned 5,620 hits.
  - “Lagrangian” returned 14,300 hits.

- In 2007:
  - “Lagrangean” returns 208,000 hits.
  - “Lagrangian” returns 5,820,000 hits.

- In 2010:
  - “Lagrangean” returns 110,000 hits (and asks “Did you mean: Lagrangian?”)
  - “Lagrangian” returns 2,610,000 hits.

- “Lagrangian” still wins!
Obtaining and Using Relaxations

- An important additional property of relaxations is that if a relaxation of \( IP \) is infeasible, then so is \( IP \).

- If the objective function of the relaxation is the same as the original, then if the optimal solution to the relaxation is feasible for \( IP \), it is optimal for \( IP \).

- The easiest way to obtain relaxations of \( IP \) is to drop some of the constraints defining the feasible set \( S \).

- We have two choices
  - LP relaxation: Drop the integrality constraints to obtain an LP.
  - Combinatorial relaxation: Drop a set of inequality constraints that make the resulting IP “easy.”

- It is “obvious” how to obtain an LP relaxation, but combinatorial relaxations are not as obvious.
Example: Traveling Salesman Problem

The TSP is a combinatorial problem \((E, \mathcal{F})\) whose ground set is the edge set of a graph \(G = (V, E)\).

- \(V\) is the set of customers.
- \(E\) is the set of travel links between the customers.

A feasible solution is a subset of \(E\) consisting of edges of the form \({i, \sigma(i)}\) for \(i \in V\), where \(\sigma\) is a simple permutation \(V\) specifying the order in which the customers are visited.

**IP Formulation:**

\[
\sum_{j=1}^{n} x_{ij} = 2 \quad \forall i \in N^- \\
\sum_{i \in S} x_{ij} \geq 2 \quad \forall S \subseteq V, |S| > 1.
\]

where \(x_{ij}\) is a binary variable indicating whether \(\sigma(i) = j\).
Combinatorial Relaxations of the TSP

• The Traveling Salesman Problem has several well-known combinatorial relaxations.

• **Assignment Problem**
  – The problem of assigning \( n \) people to \( n \) different tasks.
  – Can be solved in polynomial time.
  – Obtained by dropping the subtour elimination constraints.

• **Minimum 1-tree Problem**
  – A *1-tree* in a graph is a spanning tree of nodes \( \{2, \ldots, n\} \) plus exactly two edges incident to node one.
  – A minimum 1-tree can be found in polynomial time.
  – This relaxation is obtained by dropping all subtour elimination constraints involving node 1 and also all degree constraints not involving node 1.
Lagrangian Relaxation

• The idea is again based on relaxing a set of constraints from the original formulation.

• We try to push the solution towards feasibility by penalizing violation of the dropped constraints.

• Suppose our IP is defined by

\[
\begin{align*}
\max & \quad cx \\
\text{s.t.} & \quad A^1 x \leq b^1 \\
& \quad A^2 x \leq b^2 \\
& \quad x \in \mathbb{Z}^n_+ 
\end{align*}
\]

where optimizing over \( Q = \{ x \in \mathbb{Z}^n_+ \mid A^2 x \leq b^2 \} \) is “easy.”

• Lagrangian Relaxation:

\[
LR(\lambda) : z_{LR}(\lambda) = \max_{x \in Q} \{ (c - \lambda A^1)x + \lambda b^1 \}.
\]
Properties of the Lagrangian Relaxation

• For any $\lambda \geq 0$, $LR(\lambda)$ is a relaxation of $IP$ (why?).

• Solving $LR(\lambda)$ yields an upper bound on the value of the optimal solution.

• Because of our assumptions, $LR(\lambda)$ can be solved easily.

• Recalling LP duality, one can think of $\lambda$ as a vector of “dual variables.”

• If the solution to the relaxation is integral, it is optimal if the primal and dual solutions are complementary, as in LP.

• What is the obvious next step?