Computational Integer Programming
Universidad de los Andes

Lecture 1

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Quick Introduction

- Bio
- Course web site

  http://coral.ie.lehigh.edu/~ted/teaching/mip

- Course structure
  - Nine lectures of one hour each
  - Slides will be posted on-line
  - Computational exercises

References for This Lecture

- N&W Sections I.1.1-I.1.4
- Wolsey Chapter 1

The General Setting

- In this course, we consider mathematical programming models of the form
$$\max\{cx \mid Ax \leq b, x \in \mathbb{Z}^p_+, y \in \mathbb{R}^{n-p}_+\},$$
where $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$.
- This type of model is called a *mixed integer linear programming model*, or simply a *mixed integer program* (MIP).
- If $p = n$, then we have a *pure integer linear programming model*, or *integer program* (IP).
- The first p components of x are the *discrete* or *integer* variables and the remaining components consist of the *continuous* variables.

Some Notes

- We consider maximization problems throughout these lectures.
- I tend to think in terms of minimization by default, so please be aware, this may cause some confusion.
- Also note that all variables are assumed to be nonnegative even when not explicitly indicated.
- In most of the lectures, we will consider only the pure integer case for simplicity.
- One further assumption we will make is that the constraint matrix is rational. Why?

Solutions

- A *solution* is an assignment of values to variables.
- A solution can hence be thought of as an n-dimensional vector.
- A *feasible solution* is an assignment of values to variables such that all the constraints are satisfied.
- The *objective function value* of a solution is obtained by evaluating the objective function at the given point.
- An *optimal solution* (assuming maximization) is one whose objective function value is greater than or equal to that of all other feasible solutions.
- Note that a mathematical program may not have a feasible solution
- Question: What are the different ways in which this can happen?

Possible Outcomes

- When we say we are going to "solve" a mathematical program, we mean to determine
  - whether it is feasible, and
  - whether it has an optimal solution.
- We may also want to know some other things, such as the status of its "dual" or about sensitivity.

Special Case: Binary Integer Programs

- In many cases, the variables of an IP represent yes/no decisions or logical relationships.
- These variables naturally take on values of 0 or 1.
- Such variables are called *binary*.
- Integer programs involving only binary variables are called *binary integer programs* (BIPs).

Special Case: Combinatorial Optimization Problems

- A *combinatorial optimization problem* $CP = (N, \mathcal{F})$ consists of
  - A finite *ground set* N,
  - A set $\mathcal{F} \subseteq 2^N$ of *feasible solutions*, and
  - A *cost function* $c \in \mathbb{Z}^n$.
- The *cost* of $F \in \mathcal{F}$ is $c(F) = \sum_{j \in F} c_j$.
- The combinatorial optimization problem is then

$$\max\{c(F) \mid F \in \mathcal{F}\}$$

- Note that there is a natural association with BIPs.
- Many COPs can be written as BIPs or MIPs.

How Hard is Integer Programming?

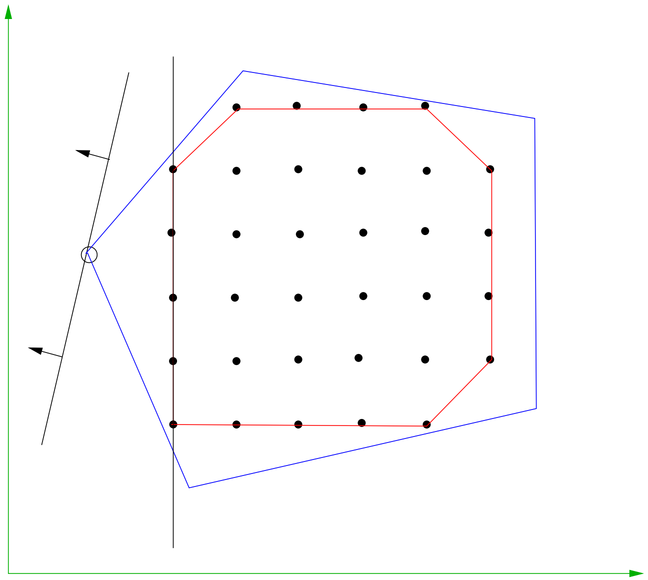
- Solving general integer programs can be much more difficult than solving linear programs.
- There in no known *polynomial-time* algorithm for solving general MIPs.
- Solving the associated *linear programming relaxation* results in an upper bound on the optimal solution to the MIP.
- In general, an optimal solution to the LP relaxation does not tell us much about an optimal solution to the MIP.
  - Rounding to a feasible integer solution may be difficult.
  - The optimal solution to the LP relaxation can be arbitrarily far away from the optimal solution to the MIP.
  - Rounding may result in a solution far from optimal.

The Geometry of Integer Programming

- Let's consider again an integer linear program

$$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & x \in \mathbb{Z}^n_+ \end{aligned}$$

- The feasible region is the integer points inside a polyhedron.



- It is easy to see why solving the LP relaxation does not necessarily yield a good solution (why?).

Dimension of Polyhedra

- The polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is of *dimension* k, denoted $dim(\mathcal{P}) = k$, if the maximum number of affinely independent points in $\mathcal{P}$ is $k+1$.
- A polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$ is *full-dimensional* if $dim(\mathcal{P}) = n$.
- Let
  - $M = \{1, \ldots, m\}$,
  - $M^= = \{i \in M \mid a_i^\top x = b_i \; \forall x \in \mathcal{P}\}$ (the *equality set*),
  - $M^\leq = M \setminus M^=$ (the *inequality set*).
- Let $(A^=, b^=), (A^\leq, b^\leq)$ be the corresponding rows of (A, b).

Proposition 1. *If $\mathcal{P} \subseteq \mathbb{R}^n$, then $dim(P) + rank(A^=, b^=) = n$*

Valid Inequalities

- The inequality denoted by (π, π_0) is called a *valid inequality* for $\mathcal{P}$ if $\pi^\top x \leq \pi_0 \; \forall x \in \mathcal{P}$.
- Note that (π, π_0) is a valid inequality if and only if $\mathcal{P}$ lies in the half-space $\{x \in \mathbb{R}^n \mid \pi^\top x \leq \pi_0\}$.
- If (π, π_0) is a valid inequality for $\mathcal{P}$ and $F = \{x \in \mathcal{P} \mid \pi^\top x = \pi_0\}$, F is called a *face* of $\mathcal{P}$ and we say that (π, π_0) *represents* or *defines* F.
- A face is said to be *proper* if $F \neq \emptyset$ and $F \neq \mathcal{P}$.
- Note that a face has multiple representations.
- The face represented by (π, π_0) is nonempty if and only if $\max\{\pi^\top x \mid x \in \mathcal{P}\} = \pi_0$.
- If the face F is nonempty, we say it *supports* $\mathcal{P}$.
- Note that the set of optimal solutions to an LP is always a face of the feasible region.

Describing Polyhedra

- If $\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, then the inequalities corresponding to the rows of $[A \mid b]$ are called a *description* of $\mathcal{P}$.
- Every polyhedron has an infinite number of descriptions.
- For obvious reasons, we would like to know the smallest possible description of a given polyhedron.
- We can drop any inequality that does not support $\mathcal{P}$, so we assume henceforth that all inequalities are supporting.

Definition 1. *If (π, π_0) and (μ, μ_0) are two valid inequalities for a polyhedron $\mathcal{P} \subseteq \mathbb{R}^n_+$, we say (π, π_0)* dominates *(μ, μ_0) if there exists $u > 0$ such that $\pi \geq u\mu$ and $\pi_0 \leq u\mu_0$.*

Definition 2. *A valid inequality (π, π_0) is* redundant *in the description of $\mathcal{P}$ if there exists a linear combination of the inequalities in the description that dominates (π, π_0).*

- We can drop redundant inequalities as well. Which ones are redundant?

Facets

Proposition 2. *Every face F of a polyhedron $\mathcal{P}$ is also a polyhedron and can be obtained by setting a specified subset of the inequalities in the description of $\mathcal{P}$ to equality.*

- Note that this result is true for any description of $\mathcal{P}$.
- This result implies that the number of faces of a polyhedron is finite.
- A face F is said to be a *facet* of $\mathcal{P}$ if $dim(F) = dim(P) - 1$.
- In fact, facets are all we need to describe polyhedra.

Proposition 3. *If F is a facet of $\mathcal{P}$, then in any description of $\mathcal{P}$, there exists some inequality representing F.*

Proposition 4. *Every inequality that represents a face that is not a facet is unnecessary in the description of $\mathcal{P}$.*

Putting It Together

Putting together what we have seen so far, we can say the following.

Theorem 1.

1. *Every full-dimensional polyhedron* $\mathcal{P}$ *has a unique (up to scalar multiplication) representation that consists of one inequality representing each facet of* $\mathcal{P}$*.*

2. *If* $dim(\mathcal{P}) = n - k$ *with* $k > 0$*, then* $\mathcal{P}$ *is described by a maximal set of linearly independent rows of* $(A^=, b^=)$*, as well as one inequality representing each facet of* $\mathcal{P}$*.*

Theorem 2. *If a facet* F *of* $\mathcal{P}$ *is represented by* (π, π_0)*, then the set of all representations of* F *is obtained by taking scalar multiples of* (π, π_0) *plus linear combinations of the equality set of* $\mathcal{P}$*.*

Formulating Integer Programs

- Just as with LP, there are many ways of describing the feasible region of an integer program.
- Unlike LP, these descriptions are usually *implicit*.
- The way in which the integer program is initially described can be extremely important computationally.
- An important component of computational integer programming are methods
- We will not be discussing formulation directly, but many of the methods we'll touch on are essentially for automatic reformulation.
- A better understanding of how solvers work should lead to an improved ability to formulate IPs.