Computational Integer Programming
Universidad de los Andes

Lecture 1

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Quick Introduction

- Bio

- Course web site

  http://coral.ie.lehigh.edu/~ted/teaching/mip

- Course structure
  - Nine lectures of one hour each
  - Slides will be posted on-line
  - Computational exercises
References for This Lecture

- N&W Sections I.1.1-I.1.4
- Wolsey Chapter 1
The General Setting

- In this course, we consider mathematical programming models of the form
  \[
  \max \{ cx \mid Ax \leq b, x \in \mathbb{Z}_+^p, y \in \mathbb{R}^{n-p} \},
  \]
  where \( A \in \mathbb{Q}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n \).

- This type of model is called a mixed integer linear programming model, or simply a mixed integer program (MIP).

- If \( p = n \), then we have a pure integer linear programming model, or integer program (IP).

- The first \( p \) components of \( x \) are the discrete or integer variables and the remaining components consist of the continuous variables.
Some Notes

- We consider \textit{maximization} problems throughout these lectures.
- I tend to think in terms of \textit{minimization} by default, so please be aware, this may cause some confusion.
- Also note that all variables are assumed to be \textit{nonnegative} even when not explicitly indicated.
- In most of the lectures, we will consider only the pure integer case for simplicity.
- One further assumption we will make is that the constraint matrix is \textit{rational}. Why?
Solutions

• A *solution* is an assignment of values to variables.

• A solution can hence be thought of as an $n$-dimensional vector.

• A *feasible solution* is an assignment of values to variables such that all the constraints are satisfied.

• The *objective function value* of a solution is obtained by evaluating the objective function at the given point.

• An *optimal solution* (assuming maximization) is one whose objective function value is greater than or equal to that of all other feasible solutions.

• Note that a mathematical program may not have a feasible solution

• **Question**: What are the different ways in which this can happen?
Possible Outcomes

• When we say we are going to “solve” a mathematical program, we mean to determine
  – whether it is feasible, and
  – whether it has an optimal solution.

• We may also want to know some other things, such as the status of its “dual” or about sensitivity.
Special Case: Binary Integer Programs

- In many cases, the variables of an IP represent yes/no decisions or logical relationships.
- These variables naturally take on values of 0 or 1.
- Such variables are called *binary*.
- Integer programs involving only binary variables are called *binary integer programs* (BIPs).
Special Case: Combinatorial Optimization Problems

- A **combinatorial optimization problem** $CP = (N, \mathcal{F})$ consists of
  - A finite **ground set** $N$,
  - A set $\mathcal{F} \subseteq 2^N$ of **feasible solutions**, and
  - A **cost function** $c \in \mathbb{Z}^n$.

- The **cost** of $F \in \mathcal{F}$ is $c(F) = \sum_{j \in F} c_j$.

- The combinatorial optimization problem is then

  $$\max\{c(F) \mid F \in \mathcal{F}\}$$

- Note that there is a natural association with BIPs.

- Many **COPs** can be written as **BIPs** or **MIPs**.
How Hard is Integer Programming?

- Solving general integer programs can be much more difficult than solving linear programs.
- There is no known \textit{polynomial-time} algorithm for solving general MIPs.
- Solving the associated \textit{linear programming relaxation} results in an upper bound on the optimal solution to the MIP.
- In general, an optimal solution to the LP relaxation does not tell us much about an optimal solution to the MIP.
  - \textbf{Rounding} to a feasible integer solution may be difficult.
  - The optimal solution to the LP relaxation can be arbitrarily far away from the optimal solution to the MIP.
  - Rounding may result in a solution far from optimal.
The Geometry of Integer Programming

• Let’s consider again an integer linear program

\[
\begin{align*}
\text{max} & \quad c^\top x \\
\text{s.t.} & \quad Ax \leq b \\
x & \in \mathbb{Z}_+^n
\end{align*}
\]

• The feasible region is the integer points inside a polyhedron.

• It is easy to see why solving the LP relaxation does not necessarily yield a good solution (why?).
Dimension of Polyhedra

• The polyhedron $\mathcal{P} = \{ x \in \mathbb{R}^n \mid Ax \leq b \}$ is of dimension $k$, denoted $\text{dim}(\mathcal{P}) = k$, if the maximum number of affinely independent points in $\mathcal{P}$ is $k + 1$.

• A polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$ is full-dimensional if $\text{dim}(\mathcal{P}) = n$.

• Let
  
  - $M = \{1, \ldots, m\}$,
  - $M^- = \{i \in M \mid a_i^T x = b_i \ \forall x \in \mathcal{P}\}$ (the equality set),
  - $M^\leq = M \setminus M^-$ (the inequality set).

• Let $(A^-, b^-), (A^\leq, b^\leq)$ be the corresponding rows of $(A, b)$.

Proposition 1. If $\mathcal{P} \subseteq \mathbb{R}^n$, then $\text{dim}(\mathcal{P}) + \text{rank}(A^-, b^-) = n$
Valid Inequalities

- The inequality denoted by $(\pi, \pi_0)$ is called a valid inequality for $\mathcal{P}$ if $\pi^\top x \leq \pi_0 \quad \forall x \in \mathcal{P}$.

- Note that $(\pi, \pi_0)$ is a valid inequality if and only if $\mathcal{P}$ lies in the half-space $\{x \in \mathbb{R}^n \mid \pi^\top x \leq \pi_0\}$.

- If $(\pi, \pi_0)$ is a valid inequality for $\mathcal{P}$ and $F = \{x \in \mathcal{P} \mid \pi^\top x = \pi_0\}$, $F$ is called a face of $\mathcal{P}$ and we say that $(\pi, \pi_0)$ represents or defines $F$.

- A face is said to be proper if $F \neq \emptyset$ and $F \neq \mathcal{P}$.

- Note that a face has multiple representations.

- The face represented by $(\pi, \pi_0)$ is nonempty if and only if $\max\{\pi^\top x \mid x \in \mathcal{P}\} = \pi_0$.

- If the face $F$ is nonempty, we say it supports $\mathcal{P}$.

- Note that the set of optimal solutions to an LP is always a face of the feasible region.
Describing Polyhedra

- If \( \mathcal{P} = \{ x \in \mathbb{R}^n \mid Ax \leq b \} \), then the inequalities corresponding to the rows of \([A \mid b]\) are called a description of \( \mathcal{P} \).

- Every polyhedron has an infinite number of descriptions.

- For obvious reasons, we would like to know the smallest possible description of a given polyhedron.

- We can drop any inequality that does not support \( \mathcal{P} \), so we assume henceforth that all inequalities are supporting.

**Definition 1.** If \((\pi, \pi_0)\) and \((\mu, \mu_0)\) are two valid inequalities for a polyhedron \( \mathcal{P} \subseteq \mathbb{R}^n_+ \), we say \((\pi, \pi_0)\) dominates \((\mu, \mu_0)\) if there exists \(u > 0\) such that \(\pi \geq u \mu\) and \(\pi_0 \leq u \mu_0\).

**Definition 2.** A valid inequality \((\pi, \pi_0)\) is redundant in the description of \( \mathcal{P} \) if there exists a linear combination of the inequalities in the description that dominates \((\pi, \pi_0)\).

- We can drop redundant inequalities as well. Which ones are redundant?
Facets

Proposition 2. Every face $F$ of a polyhedron $\mathcal{P}$ is also a polyhedron and can be obtained by setting a specified subset of the inequalities in the description of $\mathcal{P}$ to equality.

- Note that this result is true for any description of $\mathcal{P}$.
- This result implies that the number of faces of a polyhedron is finite.
- A face $F$ is said to be a facet of $\mathcal{P}$ if $\dim(F) = \dim(P) - 1$.
- In fact, facets are all we need to describe polyhedra.

Proposition 3. If $F$ is a facet of $\mathcal{P}$, then in any description of $\mathcal{P}$, there exists some inequality representing $F$.

Proposition 4. Every inequality that represents a face that is not a facet is unnecessary in the description of $\mathcal{P}$. 
Putting It Together

Putting together what we have seen so far, we can say the following.

**Theorem 1.**

1. Every full-dimensional polyhedron $\mathcal{P}$ has a unique (up to scalar multiplication) representation that consists of one inequality representing each facet of $\mathcal{P}$.

2. If $\dim(\mathcal{P}) = n - k$ with $k > 0$, then $\mathcal{P}$ is described by a maximal set of linearly independent rows of $(A^=, b^=)$, as well as one inequality representing each facet of $\mathcal{P}$.

**Theorem 2.** If a facet $F$ of $\mathcal{P}$ is represented by $(\pi, \pi_0)$, then the set of all representations of $F$ is obtained by taking scalar multiples of $(\pi, \pi_0)$ plus linear combinations of the equality set of $\mathcal{P}$.
Formulating Integer Programs

- Just as with LP, there are many ways of describing the feasible region of an integer program.
- Unlike LP, these descriptions are usually *implicit*.
- The way in which the integer program is initially described can be extremely important computationally.
- An important component of computational integer programming are methods
- We will not be discussing formulation directly, but many of the methods we’ll touch on are essentially for automatic reformulation.
- A better understanding of how solvers work should lead to an improved ability to formulate IPs.