IE 495 Lecture 21

November 14, 2000

Reading for This Lecture

- Primary
  - Miller and Boxer, Pages 124-128
  - Forsythe and Mohler, Sections 1 to 8

Matrix Multiplication

- The standard sequential algorithm for multiplying matrices is $O(n^3)$.
- Strassen's Algorithm is a divide and conquer approach.
- Analysis of Strassen's Algorithm
  - $T(n) = 7T(n/2) + dn^2$
  - $T(n) = O(n^{\log(7)}) = O(n^{2.81...})$
- Every algorithm must be $\Omega(n^2)$.
- The best known algorithm to date is $O(n^{2.376...})$.
- Can we parallelize Strassen's Algorithm?

Parallel Matrix Multiplication

- Assume a CREW shared-memory architecture with n^3 processors.
- Label processors as P_{111} through P_{nnn}.
- Processor P_{ijk} calculates $a_{ik} \cdot b_{kj}$.
- The remaining sums can be computed in *O(log n)* using a semigroup operation.
- The running time is *O(log n)*.
- Cost optimality?

Matrix Multiplication on a Mesh

- Assume a $2n \times 2n$ mesh computer.
- Assume each processor initially stores one entry.
- Algorithm
- Analysis
- Optimality

Real Vector Spaces

- A real vector space is a set $\mathcal{V}$, along with
  - an addition operation that is commutative and associative.
  - an element $0 \in V$ such that $a + 0 = a, \forall a \in \mathcal{V}$.
  - an additive inverse operation such that $\forall a \in V, \exists a' \in \mathcal{V}$ such that $a + a' = 0$.
  - a scalar multiplication operation such that $\forall \lambda, \mu \in \mathbf{R}, a, b \in \mathcal{V}$
    - $\lambda(a + b) = \lambda a + \lambda b$
    - $(\lambda + \mu)a = \lambda a + \mu a$
    - $\lambda(\mu a) = (\lambda\mu)a$
    - $1a = a$

Norms on Vector Spaces

- A norm on a vector space is a function $\|\cdot\|: \mathcal{V} \to \mathbf{R}$ satisfying
  - $\|v\| \geqslant 0 \ \forall v \in \mathcal{V}$
  - $\|v\| = 0$ if and only if $v = 0$
  - $\|v + w\| \leq \|v\| + \|w\| \ \forall v, w \in \mathcal{V}$
  - $\|\lambda v\| = |\lambda| \cdot \|v\|$
- Norms are used for measuring the "size" of an object or the "distance" between two objects in a vector space.
- These are the normal properties you would expect such a measure to have.

Examples of Vector Spaces

- $\mathbf{R}^n$
- $\mathbf{Z}^n$
- $\mathbf{R}^{n \times n}$
- $\{y \in \mathbf{R}^m : Ax = y, \exists x \in \mathbf{R}^n\}$

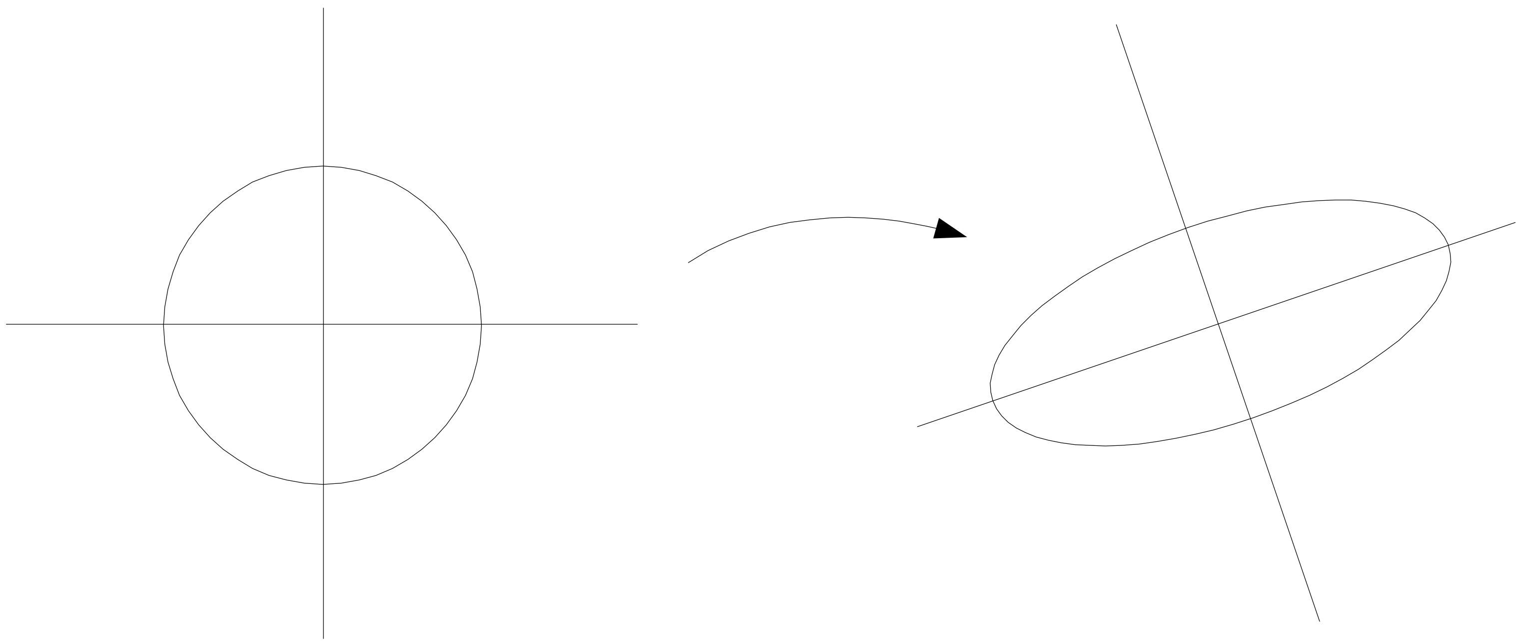
Matrix and Vector Norms

- Unless otherwise indicated, we will use the L_2 norm for vectors and the corresponding norm for matrices.
- We will denote this by $\|\cdot\|$.
- Note the following definitions and properties
  - $|x^T y| \leq \|x\| \cdot \|y\|$
  - $\|A\| = \max \{\|Ax\|/\|x\|,\ x \neq 0\}$
  - $\|Ax\| \leq \|A\| \cdot \|x\|$
  - $\|AB\| \leq \|A\| \cdot \|B\|$

Solving Systems of Equations

- Problem: Given a matrix $A \in \mathbf{R}^{n \times n}$ and a vector $b \in \boldsymbol{R}^n$, we wish to find $x \in \mathbf{R}^n$ such that $Ax = b$.
- Diagonal form of a matrix
  - An othogonal matrix U has the property the $U^TU = UU^T = \mathrm{I}$.
  - Given $A \in \mathbf{R}^{n \times n}$, there exist orthogonal matrices U, V such that
    - $U^TAV = D$ where D is a diagonal matrix where
    - diagonal elements of D are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r > \mu_{r+1} = \cdots = \mu_n = 0$, and
    - r is the rank of A.
    - μ_i is the non-negative square root of the i^{th} eigenvalue.
  - This is called the *singular value decomposition*.

Importance of the SVD



Effect of multiplying by a matrix

Implications

- Multiplying by A represents a *rotation* and a *scaling* of axes to get from one space to the other.
- μ_i is the non-negative square root of the i^{th} eigenvalue.
- Notice that $\|A\| = \|D\| = \mu_1$.
- So the norm of A is the maximum amount any axis gets magnified by A.
- If $r = n$, then we can easily derive the inverse of A.
- Also, $\|A^{-1}\| = \|A\|^{-1} = 1/\mu_n$.