Reading for This Lecture

- C&T Chapter 15
The Mortgage Market

• Mortgages represent the largest single sector of the U.S. debt market, surpassing even the federal government.

• Many financial instruments have therefore been created to provide credit to this market.

• The primary way this has been accomplished since the 1970s is the bundling together of individual mortgages into capital market instruments called mortgage-backed securities (MBSs).

• The principal and interest from the mortgages in the pool backing an MBS are passed through to investors in some fashion.

• By selling MBSs, banks can realize their fees up front and lay off their risk to the market.
Pass-through MBSs

- Initially, MBSs were simply packaged using a pass-through structure.
- Each investor received a pro rata share of principal and interest payments for mortgages in the pool.
- The problem with this approach is that the cash flows are very unpredictable due to *pre-payment risk*.
- Mortgage payers prepay for a variety of reasons, but for fixed-rate mortgages, this is usually associated with a drop in interest rates.
- This may force an unplanned reinvestment at a lower interest rate.
Collateralized Mortgage Obligations

• A collaterialized mortgage obligation is a more sophisticated MBS that rearranges cash flows to make them more predictable.

• There are many ways of doing this, but here we focus on the creation of consecutive tranches.

• The basic idea is to package the cash flows into bonds with different maturities.

• Principal payments are funneled to investors in each tranche consecutively until the obligation is repaid.


Simple Two-Tranche Model

- Suppose we have an MBS consisting of $100 million in mortgage loans.
- In a two-tranche model, we might divide the pool into two $50 million tranches.
- Initially, investors in both tranches receive interest payments, but all principal payments are funneled to the investors in the first tranche (the *fast-pay tranche*) until it is repaid.
- After the fast-pay tranche is repaid, remaining principal payments go to the second tranche.
- By restructuring in this way, the fast-pay tranche reaches maturity much earlier than the *slow-pay tranche*.
- A byproduct of the restructuring is that the risk of default is much lower for the fast-pay tranche.
- This means that the interest rate paid on the fast-pay tranche can be reduced, resulting in additional profit.
A Model of Consecutive Tranches

• Early payments are more likely to be fully funded than later ones.
• Hence, fast tranches get a higher credit rating than slower ones and can be sold at lower interest rates.
• Overall, the interest that has to be paid to buyers of the tranches is lower than the interest paid by the mortgage holders.
• Hence, the bank issuing the restructured tranches earns money.
• A bond with payback \( p_t \) of principal at time \( t \) \((t = 1, \ldots, T)\) is priced with respect to its weighted average life (WAL)

\[
WAL = \frac{\sum_{t=1}^{T} tp_t}{\sum_{t=1}^{T} p_t}.
\]

• A bond with a WAL of \( n \) years will be priced like a treasury bond with a duration of \( n \) years plus a spread (extra interest) which depends on the credit rating.
Credit Ratings and Spreads

• The spot rates of future coupon payments and the spreads for different credit ratings can be looked up in a table.

• For example, the table could look as follows, where the spreads under the credit ratings (AAA etc.) are given in basis points, i.e., in \( \frac{1}{100} \) of 1%.

<table>
<thead>
<tr>
<th>duration</th>
<th>spot rate</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.74%</td>
<td>85</td>
<td>100</td>
<td>115</td>
<td>130</td>
<td>165</td>
<td>220</td>
<td>345</td>
</tr>
<tr>
<td>2</td>
<td>4.89%</td>
<td>90</td>
<td>105</td>
<td>125</td>
<td>140</td>
<td>190</td>
<td>275</td>
<td>425</td>
</tr>
<tr>
<td>3</td>
<td>5.05%</td>
<td>95</td>
<td>110</td>
<td>135</td>
<td>150</td>
<td>210</td>
<td>335</td>
<td>500</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
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</tr>
</tbody>
</table>
Notation

- $Q_0$ is the amount of principal to be repaid.
- $T$ is the horizon over which the principal is to be repaid.
- $I_t$ is the interest to paid in year $t$.
- $A_t$ is the scheduled amortization payment in year $t$.
- $q_t$ is the pre-payment rate in year $t$.
- $R_t$ is the pre-payment amount in year $t$.
- $P_t$ is the total payment in year $t$ (including pre-payment).
- $Q_t$ is the amount of principal outstanding at the end of year $t$.
- $r \times 100\%$ is the compound yearly interest rate.
Example

Let’s take $Q_0 = 100$, $r = 0.1$, $T = 10$ $q_1 = .01\%$. If the principal is to be repaid in equal installments, then the scheduled amortization payment in year 1 is

$$A_1 = \frac{Q_0r}{(1+r)^T - 1} = 6.27.$$ 

So we have for year 1:

- **Interest**: $I_1 = rQ_0 = 10$.
- **Scheduled amortization**: $A_1 = \frac{Q_0r}{(1+r)^T - 1} = 6.27$.
- **Prepayment**: $R_1 = q_1(Q_0 - 6.27) = 0.937$.
- **Total principal pay down**: $P_1 = R_1 + A_1 = 6.27 + 0.937 = 7.207$.
- **Principal left after year 1**: $Q_1 = Q_0 - P_1 = 92.793$. 
Generalizing

In general, we have a given scenario $q_1, \ldots, q_T$ of prepayment rates in years $1, 2, \ldots, T$. In year $t$, we have

- **Interest**: $I_t = rQ_{k-1}$.
- **Scheduled amortization**: $A_t = Q_{k-1}r/[(1 + r)^T - 1]$.
- **Prepayment**: $R_t = q_t(Q_{k-1} - A_t)$.
- **Total principal pay down**: $P_t = R_t + A_t$.
- **Principal left after year 1**: $Q_t = Q_{k-1} - P_t$.

One can thus recursively compute the corresponding $(I_t, P_t, Q_t)$ ($t = 1, \ldots, T$).
Packaging

• The model we have presented is a simplification of the real problem.

• In real CMOs, the pay-back time of the principal is also variable, rather than just the amount of principal paid back.

• Nevertheless, this model is a good approximation to the real one and leads to very similar results.

• Once the payouts $P_t$ are known, the question is how to optimally package them into consecutive tranches

\[(P_1, \ldots, P_{T_1}), (P_{T_1+1}, \ldots, P_{T_2}), \ldots, (P_{\ldots}, \ldots, P_T).\]
**Candidate Tranches**

- Let us refer to the candidate tranche \((P_j, \ldots, P_t)\) as \((j, t)\).
- Associated with the candidate tranche \((j, t)\) is its *buffer*

\[
B_{jt} = \frac{\sum_{k=t+1}^{T} P_k}{\sum_{k=1}^{T} P_k},
\]

- The buffer is the proportion of principal left after the tranche expires.
- Each tranche also has its own WAL

\[
WAL_{jt} = \frac{\sum_{k=j}^{t} kP_k}{\sum_{k=j}^{t} P_k}.
\]

- Note that this is the WAL of a bond that has no repayment of principal for the first \(j - 1\) years, but interest (coupons) is still paid during this time.
Prepayment Scenarios

• In order to achieve a high quality ranking, a tranche must be able to sustain higher than expected default rates without compromizing payments to the tranche holders.

• The default rate is determined by the scenario of prepayment rates \( q_1, \ldots, q_T \).

• Regulatory bodies require that several prescribed scenarios be tested.

• For example, the Public Securities Association (PSA) industry standard benchmark is \( q_1 = .01, q_2 = .03, q_3 = .05 \), and \( q_t = .06 \) for \( t \geq 4 \).
Tranche Credit Ratings

• For a tranche to be given a certain credit rating, it must satisfy

\[ B_{jt} \geq WAL_{jt} \cdot d \cdot L, \]

where \( L \) is the *loss multiple*, specified as follows,

<table>
<thead>
<tr>
<th>Rating</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1.5</td>
<td>0</td>
</tr>
</tbody>
</table>

• Hence, the earlier tranches naturally receive higher credit ratings.
Present Value of a Tranche

• From \((B_{jt}, W_{jt})\) and the above table one can thus compute the credit rating for each candidate tranche \((j, t)\).

• This rating implies a coupon rate \(c_{jt}\) that can be read off the earlier table of spot rates and spreads.

• Using the coupon rates \(c_{jt}\), the net present value \(Z_{jt}\) of tranche \((j, t)\) can be computed:

  – In period \(k\), a payment of \(c_{jt}\) times the remaining principal on the tranche is paid (as interest), and if \(k \in [j, t]\), then the principal payment \(P_k\) is made.
  – The result is a total payment of \(C_k\).
  – Then the present value is \(T_{jt} = \sum_{k=1}^{t} C_k/(1 + r_k)^k\).
A Dynamic Programming Formulation

- To maximize earnings, the issuer now wants to structure the CMO into $K$ sequential tranches so as to maximize the net present value of total payments to bond-holders.
- This enables the entire mortgage portfolio to be sold off at the maximum price.
- The stages will be the number of tranches and the states will be the years $1, \ldots, T$.
- We set the value function to be

$$v(k, t) = \text{The maximum present value of total payments to bondholders in years 1 through } t$$
when the CMO has $k$ tranches up to year $t$.

- Then

$$v(k, t) = \max_{j=k-1, \ldots, t-1} \{v(k - 1, j) + T_{j+1, t}\}.$$
Finding the Optimal CMO Structure

• Using this recurrence, we compute $v(k, t)$ for $k = 1, \ldots, K$ and $t = 1, \ldots, T$.

• The optimal net present value of future payments to bond holder is then $v(K, T)$. 