Reading for This Lecture

- C&T Chapter 1
What will this class be about?

- **Modeling** financial optimization problems.
- Interpreting those models as **mathematical programs**.
- Analyzing those mathematical programs using optimization **methodology** and **software**.
- Using the analysis to **gain insight** and **make decisions**.
What is the purpose of modeling?

• The exercise of building a model can provide insight.
• It’s possible to do things with models that we can’t do with “the real thing.”
• Analyzing models can help us decide on a course of action.
The Modeling Process

• The modeling process consists generally of the following steps.
  – Determine the “real-world” state variables, system constraints, and goal(s) or objective(s) for operating the system.
  – Translate these variables and constraints into the form of a mathematical optimization problem (the “formulation”).
  – Solve the mathematical optimization problem.
  – Interpret the solution in terms of the real-world system.

• This process presents many challenges.
  – Simplifications may be required in order to ensure the eventual mathematical program is “tractable”.
  – The mappings from the real-world system to the model and back are sometimes not very obvious.
  – There may be more than one valid “formulation”.

• All in all, an intimate knowledge of mathematical optimization definitely helps during the modeling process.
Types of (Mathematical) Models

- There are many different classes of mathematical models that may be used to analyze financial decision-making and prediction problems.
  - Simulation Models
  - Probability Models
  - Stochastic Models
  - Behavior Models
  - Mathematical Programming/Optimization Models

- Mathematical optimization models are widely used in practice and we will focus only on these.

- This will provide a rich class of methods for analysis.
Mathematical Programming/Optimization Models

- What does *mathematical programming* mean?
- Programming here means “planning.”
- Literally, these are “mathematical models for planning.”
- Also called *optimization models*.
- A *mathematical optimization problem* consists of
  - a set of *variables* that describe the state of the system,
  - a set of *constraints* that determine the states that are allowable,
  - external input *parameters* and *data*, and
  - an *objective function* that provides an assessment of how well the system is functioning.
- The variables represent operating *decisions* that must be made.
- The constraints represent operating *specifications*.
- The goal is to determine the *best* operating state consistent with specifications.
Forming a Mathematical Optimization Model

The general form of a mathematical optimization model is:

\[
\begin{align*}
\min \text{ or } \max & \quad f(x_1, \ldots, x_n) \\
\text{s.t.} & \quad g_i(x_1, \ldots, x_n) \begin{cases} \leq \\ = \\ \geq \end{cases} b_i, \ i \in M \\
& \quad (x_1, \ldots, x_n) \in X
\end{align*}
\]

\(X\) may be a discrete set, such as \(\mathbb{Z}^n\).

Notes:

- There is an important assumption here that all input data are known and fixed.
- Such a mathematical program is called deterministic.
- Is this realistic?
Types of Mathematical Optimization Problema

• Unless otherwise specified, optimization problems are usually assumed to be deterministic.

• The type of a (deterministic) optimization problem is determined primarily by
  – The form of the objective and the constraints.
  – The set $X$.

• A wide range of mathematical optimization model types are described at
  – the NEOS Guide, and
  – the on-line version of *Operations Research Models and Methods*.

• We will review the various classes in more detail in Lecture 2.
Solutions

• What is the result of analyzing an optimization problem?

• A solution is an assignment of values to variables.

• A solution can be thought of as a vector.

• A feasible solution is an assignment of values to variables such that all the constraints are satisfied.

• The objective function value of a solution is obtained by evaluating the objective function at the given solution.

• An optimal solution (assuming minimization) is one whose objective function value is less than or equal to that of all other feasible solutions.

• We may also be interested in some additional qualities of the solution.
  – Sensitivity
  – Robustness
  – Risk
Stochastic Optimization

- In the real world, little is known ahead of time with certainty.
- A *risky investment* is one whose return is not known ahead of time.
- a *risk-free* investment is one whose return is fixed.
- To make decision involving risky investments, we need to incorporate some degree of *stochasticity* into our models.
Types of Uncertainty

• Where does uncertainty come from?
  – Weather
  – Financial Uncertainty
  – Market Uncertainty
  – Competition
  – Technology

• In decision analysis, we must proceed through this list and identify items that might affect the outcome of a decision.
The Scenario Approach to Uncertainty

- The scenario approach assumes that there are a finite number of possible future outcomes of uncertainty.

- Each of these possible outcomes is called a scenario.
  - Demand for a product is “low, medium, or high.”
  - Weather is “dry or wet.”
  - The market will go “up or down.”

- Even if this is not reality, often a discrete approximation is useful.
Multi-period Optimization Models

• When we introduce time as an element of a stochastic optimization model, we also have to address the following questions.
  – When do we have to make a given decision?
  – What will we know at the time we are making the decision?
  – How far into the future are we looking?

• Multi-stage models generally assume that decision are made in stages and that in each stage, some amount of uncertainty is resolved.

• Example
  – Fred decides to rebalance his investment portfolio once a quarter.
  – At the outset, he only knows current prices and perhaps some predictions about future prices.
  – At the beginning of each quarter, prices have been revealed and Fred gets a chance to make a “recourse decision.”
Example 1: Short Term Financing

A company needs to make provisions for the following cash flows over the coming five months: $-150K$, $-100K$, $200K$, $-200K$, $300K$.

- The following sources of funds are available,
  - Up to $100K$ credit at $1\%$ interest per month,
  - The company can issue a 2-month zero-coupon bond yielding $2\%$ interest over the two months,
  - Excess funds can be invested at $0.3\%$ monthly interest.

- How should the company finance these cash flows if no payment obligations are to remain at the end of the period?
Example 1 (cont.)

- Note that all investments are risk-free.

- What are the decision variables?
  - $x_i$, the amount drawn from the line of credit in month $i$,
  - $y_i$, the number of bonds issued in month $i$,
  - $z_i$, the amount invested in month $i$,
  - $v$, the wealth of the company at the end of Month 5.
Example 1 (cont.)

The problem can then be modelled as the following linear program:

\[
\begin{align*}
\max_{(x,y,z,v) \in \mathbb{R}^{12}} \quad & f(x, y, z, v) = v \\
\text{s.t.} \quad & x_1 + y_1 - z_1 = 150 \\
& x_2 - 1.01x_1 + y_2 - z_2 + 1.003z_1 = 100 \\
& x_3 - 1.01x_2 + y_3 - 1.02y_1 - z_3 + 1.003z_2 = -200 \\
& x_4 - 1.01x_3 - 1.02y_2 - z_4 + 1.003z_3 = 200 \\
& \quad - 1.01x_4 - 1.02y_3 - v + 1.003z_4 = -300 \\
& 100 - x_i \geq 0 \quad (i = 1, \ldots, 4) \\
& x_i \geq 0 \quad (i = 1, \ldots, 4) \\
& y_i \geq 0 \quad (i = 1, \ldots, 3) \\
& z_i \geq 0 \quad (i = 1, \ldots, 4) \\
& v \geq 0.
\end{align*}
\]
Example 2: Portfolio Optimization

• We have the choice to invest in either Intel, Wal-Mart, or G.E..
• All stocks currently cost $100/share.
• In one year’s time, the market will be either “up” or “down.”
• the Following are the possible outcomes

<table>
<thead>
<tr>
<th></th>
<th>Intel</th>
<th>Pepsico</th>
<th>Wal-Mart</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>130</td>
<td>112</td>
<td>115</td>
</tr>
<tr>
<td>Down</td>
<td>90</td>
<td>108</td>
<td>103</td>
</tr>
</tbody>
</table>

• What would you do?
Example 2 (cont.)

• If the probability that the market is up is \( p \), then the value at year 1 from investing $1000 are as follows:
  
  – Intel: \( 900 + 400p \)
  – Pepsico: \( 1080 + 40p \)
  – Wal-Mart: \( 1030 + 120p \)

• What is your objective?
Convex Sets

A set $S$ is convex

$$x_1, x_2 \in S, \lambda \in [0, 1] \Rightarrow \lambda x_1 + (1 - \lambda)x_2 \in S$$

• If $y = \sum \lambda_i x_i$, where $\lambda_i \geq 0$ and $\sum \lambda_i = 1$, then $y$ is a convex combination of the $x_i$’s.

• If the positivity restriction on $\lambda$ is removed, then $y$ is an affine combination of the $x_i$’s.

• If we further remove the restriction that $\sum \lambda_i = 1$, then we have a linear combination.
Example: Convex and Nonconvex Sets

CONVEX

NONCONVEX
**Review: Convex Functions**

**Definition 1.** Let $S$ be a nonempty convex set on $\mathbb{R}^n$. Then the function $f : S \to \mathbb{R}$ is said to be **convex** on $S$ if

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

for each $x_1, x_2 \in S$ and $\lambda \in (0, 1)$.

- **Strictly convex** means the inequality is strict.
- **(Strictly) concave** is defined analogously.
- A function that is neither convex nor concave is called **nonconvex**.
- Can a function be both concave and convex?
Example: Convex and Nonconvex Functions

CONVEX

NONCONVEX
The Epigraph

Convex sets and convex functions are related by the following result:

**Definition 2.** Let $S$ be a nonempty set on $\mathbb{R}^n$ and let $f : S \rightarrow \mathbb{R}$. The epigraph of $f$ is a subset of $\mathbb{R}^{n+1}$ defined by

$$\text{epi}f = \{(x, y) : x \in S, y \in \mathbb{R}, y \geq f(x)\}$$

**Theorem 1.** Let $S$ be a nonempty convex set on $\mathbb{R}^n$ and let $f : S \rightarrow \mathbb{R}$. $f$ is convex if and only if $\text{epi}f$ is a convex set.
Review: Local versus Global Optimization

- For a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, a local minimizer is an $\hat{x} \in \mathbb{R}^n$ such that $f(\hat{x}) \leq f(x)$ for all $x$ in a neighborhood of $\hat{x}$.

- In general, it is “easy” to find local minimizers.

- A global minimizer is $x^* \in \mathbb{R}^n$ such that $f(x^*) \leq f(x)$ for all $x \in \mathbb{R}$.

- The importance of convexity is the following:

**Theorem 2.** For a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, if $x^* \in \mathbb{R}^n$ is a local optimal solution to $\min_{x \in \mathbb{R}^n} f(x)$, then $x^* \in \mathbb{R}^n$ is also a global optimal solution.
Review: Vectors and Matrices

- An $m \times n$ matrix is an array of $mn$ real numbers:

$$A = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}$$

- $A$ is said to have $n$ columns and $m$ rows.

- An $n$-dimensional column vector is a matrix with one column.

- An $n$-dimensional row vector is a matrix with one row.

- By default, a vector will be considered a column vector.

- The set of all $n$-dimensional vectors will be denoted $\mathbb{R}^n$.

- The set of all $m \times n$ matrices will be denoted $\mathbb{R}^{m \times n}$. 
Review: Matrices

• The *transpose* of a matrix $A$ is

\[
A^\top = \begin{bmatrix}
a_{11} & a_{21} & \cdots & a_{m1} \\
a_{12} & a_{22} & \cdots & a_{m2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1n} & a_{2n} & \cdots & a_{mn}
\end{bmatrix}
\]

• If $x, y \in \mathbb{R}^n$, then $x^\top y = \sum_{i=1}^{n} x_i y_i$.

• This is called the *inner product* of $x$ and $y$.

• If $A \in \mathbb{R}^{m \times n}$, then $A_j$ is the $j^{th}$ column, and $a_j^\top$ is the $j^{th}$ row.

• If $A \in \mathbb{R}^{m \times k}, B \in \mathbb{R}^{k \times n}$, then $[AB]_{ij} = a_i^\top B_j$. 
Review: Linear Functions

- A linear function $f : \mathbb{R}^n \to \mathbb{R}$ is a weighted sum, written as

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} c_i x_i$$

for given coefficients $c_1, \ldots, c_n$.

- We can write $x_1, \ldots, x_n$ and $c_1, \ldots, c_n$ as vectors $x, c \in \mathbb{R}^n$ to obtain:

$$f(x) = c^\top x$$

- In this way, a linear function can be represented simply as a vector.