Integer Programming
ISE 418

Lecture 5

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Reading for This Lecture

- N&W Sections I.4.4 and I.4.6
- Wolsey Section 9.1
Describing Polyhedra

In Lecture 4, we derived the following fundamental results.

**Theorem 1.**

1. Every full-dimensional polyhedron $\mathcal{P}$ has a unique (up to scalar multiplication) representation that consists of one inequality representing each facet of $\mathcal{P}$.

2. If $\dim(\mathcal{P}) = n - k$ with $k > 0$, then $\mathcal{P}$ is described by a maximal set of linearly independent rows of $(A^=, b^=)$, as well as one inequality representing each facet of $\mathcal{P}$.

**Theorem 2.** If a facet $F$ of $\mathcal{P}$ is represented by $(\pi, \pi_0)$, then the set of all representations of $F$ is obtained by taking scalar multiples of $(\pi, \pi_0)$ plus linear combinations of the equality set of $\mathcal{P}$.

For the remainder of this lecture, let $\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ for $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m$. 
Extreme Points

Definition 1. \( x \) is an extreme point of \( P \) if there do not exist \( x^1, x^2 \in P \) such that \( x = \frac{1}{2}x^1 + \frac{1}{2}x^2 \).

Proposition 1. \( x \) is an extreme point of \( P \) if and only if \( x \) is a zero-dimensional face of \( P \).

Proposition 2. If \( P \neq \emptyset \) and \( \text{rank}(A) = n - k \), then \( P \) has a face of dimension \( k \) and no proper face of lower dimension.

- These three results together imply that \( P \) has an extreme point if and only if \( \text{rank}(A) = n \).

- This is the case for any polytope or any polyhedron lying in the nonnegative orthant.

- Recall that in 406, we showed that a polyhedron has an extreme point if and only if it does not contain a line.

- Don’t confuse \( \text{rank}(A) = n \) with \( P \) being full-dimensional!
Extreme Rays

**Definition 2.** The recession cone $\mathcal{P}^0$ associated with $\mathcal{P}$ is $\{r \in \mathbb{R}^n | Ar \geq 0\}$. Members of the recession cone are called rays of $\mathcal{P}$.

**Definition 3.** $r$ is an extreme ray of $\mathcal{P}$ if there do not exist rays $r^1$ and $r^2$ of $\mathcal{P}$ such that $r = \frac{1}{2}r^1 + \frac{1}{2}r^2$.

**Proposition 3.** If $\mathcal{P} \neq \emptyset$, then $r$ is an extreme ray of $\mathcal{P}$ if and only if $\{\lambda r | \lambda \in \mathbb{R}_+\}$ is a one-dimensional face of the recession cone.

- Note that if $r$ is an extreme ray, then so is $\lambda r$ for $\lambda > 0$.
- We need only consider one “representative” of each one-dimensional face of the recession cone.
- We can do this by choosing extreme rays $r$ with $||r|| = 1$.
- The last two results together imply that a polyhedron has a finite number of extreme points and extreme rays.
Some Results from Linear Optimization

**Theorem 3.** If $\mathcal{P} \neq \emptyset$, $\text{rank}(A) = n$, and $\max\{cx \mid x \in \mathcal{P}\}$ is finite, then there is an optimal solution that is an extreme point.

**Theorem 4.** For a given extreme point $x^*$, there exists a $c \in \mathbb{Z}^n$ such that $x^*$ is the optimal solution to $\max\{cx \mid x \in \mathcal{P}\}$.

**Theorem 5.** If $\mathcal{P} \neq \emptyset$, $\text{rank}(A) = n$, and $\max\{cx \mid x \in \mathcal{P}\}$ is unbounded, then there is an extreme ray $r^*$ with $cr^* > 0$.

- Note again that the set of all optimal solutions to a linear optimization problem is a face of the associated polyhedron.
- We call this the *optimal face*.
- Combining these results, we get Minkowski’s Theorem.
Minkowski’s Theorem

**Theorem 6.** If $\mathcal{P} \neq \emptyset$ and $\text{rank}(A) = n$, then

$$
\mathcal{P} = \left\{ \sum_{k \in K} \lambda_k x_k^k + \sum_{j \in J} \mu_j r^j \mid \lambda_k \geq 0 \text{ for } k \in K, \mu_j \geq 0 \text{ for } j \in J, \sum_{k \in K} \lambda_k = 1 \right\}.
$$

where $\{x^k\}_{k \in K}$ are the extreme points and $\{r^j\}_{j \in J}$ are the (representative) extreme rays.

**Corollary 1.** A nonempty polyhedron is bounded if and only if it has no extreme rays.

**Corollary 2.** A polytope is the convex hull of its extreme points.

- A set of the form given above is called *finitely generated* when $J$ and $K$ are finite sets.
- When $J$ or $K$ is not finite, then $\mathcal{P}$ is the feasible region of a *semi-infinite optimization problem*.
- This result is often stated as “every polyhedron is finitely generated.”
More Results from Linear Optimization

Define the following:

- \( \mathcal{P} = \{ x \in \mathbb{R}^n_+ \mid Ax \leq b \} \), \( z = \max \{ cx \mid x \in \mathcal{P} \} \)
- \( \mathcal{Q} = \{ u \in \mathbb{R}^m_+ \mid uA \geq c \} \), \( w = \min \{ ub \mid u \in \mathcal{Q} \} \)
- \( \{ x^k \}_{k \in K} \), \( \{ u^i \}_{i \in I} \) are the extreme points of \( \mathcal{P} \) and \( \mathcal{Q} \) respectively.
- \( \{ r^j \}_{j \in J} \), \( \{ v^t \}_{t \in T} \) are the extreme rays of \( \mathcal{P}^0 \) and \( \mathcal{Q}^0 \) respectively.

**Theorem 7.** \( \mathcal{P} \neq \emptyset \Leftrightarrow v^t b \geq 0 \ \forall t \in T \)

**Theorem 8.** The following are equivalent when \( \mathcal{P} \neq \emptyset \):

1. \( z \) is unbounded from above;
2. there exists an extreme ray \( r^j \) of \( \mathcal{P} \) with \( cr^j > 0 \); and
3. \( \mathcal{Q} = \emptyset \).

**Theorem 9.** If \( \mathcal{P} \neq \emptyset \) and \( z \) is bounded, then

\[
    z = \max_{k \in K} cx^k = w = \min_{i \in I} u^i b
\]
The Projection of a Polyhedron

- We will often be interested in “projecting out” a set of variables, i.e., projecting $\mathcal{P}$ into a subspace $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p \mid y = 0\}$.
- The projection of a point $(x, y)$ into this subspace is the point $(x, 0)$.
- Let $\mathcal{P} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p \mid Ax + Gy \leq b\}$
- So the projection of $\mathcal{P}$ into the space of just the $x$ variables is

$$\text{proj}_x(\mathcal{P}) = \{x \in \mathbb{R}^n \mid (x, y) \in \mathcal{P}\}$$

$$= \{x \in \mathbb{R}^n \mid v^t(b - Ax) \geq 0 \forall t \in T\}$$

where $\{v^t\}_{t \in T}$ are the extreme rays of $Q = \{v \in \mathbb{R}_+^m \mid vG = 0\}$.
- This immediately implies that the projection of a polyhedron is a polyhedron.
Weyl’s Theorem

**Theorem 10.** If

\[
Q = \left\{ \sum_{k \in K} \lambda_k x_k^k + \sum_{j \in J} \mu_j r^j \mid \lambda_k \geq 0 \text{ for } k \in K, \mu_j \geq 0 \text{ for } j \in J, \sum_{k \in K} \lambda_i = 1 \right\},
\]

where \( \{x_k^k\}_{k \in K} \) and \( \{r^j\}_{j \in J} \) are given sets of rational vectors, then \( Q \) is a rational polyhedron.

- This is the converse of Minkowski’s Theorem.
- This says roughly “every finitely generated set is a polyhedron” (remember the rationality assumption).
- The proof is easy using projection.
The Fundamental Theorem

• We have already discussed informally the fact that an integer optimization problem can, in theory, be reduced to a linear optimization problem.
• We now make these ideas more formal.
• To do so, we would now like to show the following:

**Theorem 11.** (The Fundamental Theorem of Integer Optimization) If \( \mathcal{P} = \{ x \in \mathbb{R}^n \mid Ax \leq b \} \), where \( A \in \mathbb{Q}^{m\times n} \), \( b \in \mathbb{Q}^m \), and \( S = \mathcal{P} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p}) \), then \( \text{conv}(S) \) is a rational polyhedron with the same recession cone as \( \mathcal{P} \).
Proving $\mathcal{S}$ Is Finitely Generated

- This result is easily proven if $\mathcal{S}$ is bounded (how?).
- If $\mathcal{S}$ is not bounded, then it is not so obvious.
- Our approach will be to show that $\mathcal{S}$ itself can be finitely generated.
- It then follows that $\text{conv}(\mathcal{S})$ is finitely generated.
Proving $\mathcal{S}$ Is Finitely Generated (cont.)

• Consider $\mathcal{P}$ and $\mathcal{S}$ from Theorem 11.

• By Minkowski’s Theorem, we can write

$$
\mathcal{P} = \left\{ \sum_{k \in K} \lambda_k x^k + \sum_{j \in J} \mu_j r^j \mid \lambda_k \geq 0 \text{ for } k \in K, \mu_j \geq 0 \text{ for } j \in J, \sum_{k \in K} \lambda_k = 1 \right\},
$$

with $\{x^k\}_{k \in K}$ the extreme points and $\{r^j\}_{j \in J}$ the extreme rays.

• We can assume \textit{wlog} that the extreme rays are integral.

• Then $\mathcal{S}$ is finitely generated by $\mathcal{Q} \cap \mathbb{Z}^n$ and the extreme rays of $\mathcal{P}$, where

$$
\mathcal{Q} = \left\{ \sum_{k \in K} \lambda_k x^k + \sum_{j \in J} \mu_j r^j \mid \lambda_k \geq 0 \text{ for } k \in K, 0 \leq \mu_j < 1 \text{ for } j \in J, \sum_{k \in K} \lambda_k = 1 \right\},
$$
Example

• Let’s find a finite set of generators for the set $S = \mathcal{P} \cap \mathbb{Z}^2$, where

\[
\mathcal{P} = \{ x \in \mathbb{R}^2_+ \mid 5x_1 + 3x_2 \geq 10, 5x_1 - 5x_2 \geq -1, -x_1 + 2x_2 \geq -2 \}
\]

• The generators for $S$ are the set of integer points inside the set $Q$ defined previously.

• Set $\mathcal{P}$ and its generator are shown in Figure 1 on the next slide.

• The set $Q$ is defined as

\[
Q = \{ \lambda_1 e_1 + \lambda_2 e_2 + \mu_1 r_1 + \mu_2 r_2 \mid \lambda_1, \lambda_2 \in \mathbb{R}_+, \lambda_1 + \lambda_2 = 1, \mu_1, \mu_2 \in [0, 1) \}
\]

• The generators for $S$ itself are then the points

\[
\{(2, 0), (2, 1), (2, 2), (3, 1), (3, 2), \text{ and } (4, 1)\},
\]

along with the extreme rays $(1, 1)$ and $(2, 1)$ of the recession cone.

• In this case, just the points $(2, 0), (2, 1), (2, 2)$ are a minimal set of generators, since the other points above can be generated by those.
Example

Figure 1: Generators for $\mathcal{P}$, the convex hull of $S$, and $Q$. 
Consequences

- Once we have that $S$ is finitely generated then we can easily show that $\text{conv}(S)$ is a rational polyhedron.

- Note that this result extends easily to the mixed case with rational data.

- Note also that if $\mathcal{P} \cap \mathbb{Z}^n \neq \emptyset$, then the extreme rays of $\mathcal{P}$ and $\text{conv}(S)$ coincide.

- This also shows that solving the IP $\max\{cx | x \in S\}$ is essentially equivalent to solving the LP $\max\{cx | x \in \text{conv}(S)\}$.
  
  - The objective function of the IP is unbounded if and only if the objective function of the LP is unbounded.
  - If the LP has a bounded optimal value, then it has an optimal solution that is an optimal solution to the IP (an extreme point of $\text{conv}(S)$).
  - if $\hat{x}$ is an optimal solution to IP, then it is an optimal solution to the LP.

- We can also show that an IP is either infeasible, unbounded, or has an optimal solution.
**Implicitly Described Polyhedra**

- $\text{conv}(S)$ is an “implicitly defined” polyhedron in the sense that we do not generally have a description of it in terms of half-spaces or generators.

- Knowing that $\text{conv}(S)$ is a polyhedron does not help much in obtaining an explicit description of it.

- It will, however, help in proving convergence of solution methods and in other important ways.

- In some case, we will try to generate parts of the description of this polyhedron.

- Not all the inequalities appearing in the formulation will be facet-defining for it.

- Using the properties of polyhedra that we know, we will try to determine which inequalities from the formulation are the facet-defining ones.

- We will also try to generate new valid inequalities that are facet-defining.

- Adding these to the formulation will necessarily increase its “strength.”