Integer Programming
ISE 418

Lecture 30

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Reading for This Lecture

- Nemhauser and Wolsey Sections III.3.1-III.3.3, II.1.1
- Wolsey Chapter 3
- CCZ Chapter 4

Combinatorial Algorithms for Integer Programming

- Some integer programs can be solved using so-called *combinatorial algorithms*, which do not rely on polyhedral theory.
- For example, there are various combinatorial algorithms for solving minimum cost network flow problem and special cases.
  - Shortest Path Problem
  - Maximum Flow Problem
  - Matching Problem
- Another related example is the Minimum Cut Problem.
- Algorithms for these problems are covered in more detail in a network flows course.

The Maximum Spanning Tree Problem

- Consider an undirected graph $G = (N, E)$ with cost vector $c \in \mathbb{Z}^E$.
- The *Maximum Spanning Tree Problem* is to find a spanning tree of maximum total cost.
- Such a spanning tree can be found using a *greedy algorithm*.
- Greedy Algorithm
  1. Order the edges in nonincreasing weight order, so that $c_1 \geq c_2 \geq \cdots \geq c_m$ where c_t is the cost of edge e_t.
  2. Start with the graph $G_0 = (V_0, E_0)$ consisting of a single node.
  3. At step k, if the graph $G_{k-1} \cup \{e_k\}$ contains no cycle, then set $G_k \leftarrow G_{k-1} \cup \{e_k\}$. Otherwise, set $G_k \leftarrow G_{k-1}$.
- Note that we can stop whenever the number of nodes reaches $n - 1$.
- This algorithm is guaranteed to give the optimal solution for this problem.
- Does the greedy algorithm work for other problems?

Submodular Functions

Definition 1. *A set function $f : 2^N \to \mathbb{R}$ is submodular if*

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B) \text{ for all } A, B \subseteq N.$$

Definition 2. *A set function f is nondecreasing if*

$$f(A) \leq f(B) \text{ for all } A, B \text{ with } A \subset B \subseteq N.$$

Proposition 1. *A set function f is nondecreasing and submodular if and only if*

$$f(A) \leq f(B) + \sum_{j \in A \setminus B} \left[f(B \cup \{j\}) - f(B) \right].$$

Submodular Polyhedra

- We now consider a *submodular polyhedron* defined by

$$\mathcal{P}(f) = \{x \in \mathbb{R}^n_+ \mid \sum_{j \in S} x_j \leq f(S) \text{ for } S \subseteq N\}.$$

- We are interested in solving the associated submodular optimization problem

$$\max\{cx : x \in \mathcal{P}(f)\}$$

- Consider the following greedy algorithm.
  - Order the variables so that $c_1 \geq c_2 \geq \cdots \geq c_r > 0 \geq c_{r+1} \geq \cdots \geq c_n$.
  - Set $x_i = f(S^i) - f(S^{i-1})$ for $i = 1, \ldots, r$ and $x_j = 0$ for $j > r$, where $S^i = \{1, \ldots, i\}$ for $i = 1, \ldots, r$ and $S^0 = \emptyset$.

The Greedy Algorithm and Matroids

- Surprisingly, the greedy algorithm solves all submodular optimization problems!
- Furthermore, when f is integer-valued, the greedy algorithm provides an integral solution.
- This follows from the fact the associated submodular polyhedron is TDI.
- In the special case when $f(S \cup \{j\}) - f(S) \in \{0, 1\}$, we call f a *submodular rank function*.

  Definition 3. *Given a submodular rank function r, a set $A \subseteq N$ is* independent *if $r(A) = |A|$. The pair $(N, \mathcal{F})$, where $\mathcal{F}$ is the set of independent sets is called a* matroid*.*

Properties of Matroids

- Given a matroid $(N, \mathcal{F})$.
  1. If A is an independent set and $B \subseteq A$, then B is an independent set.
  2. If A and B are independent sets with $|A| > |B|$, then there exists some $j \in A \setminus B$ such that $A \cup \{j\}$ is independent.
  3. Every maximal independent set has the same cardinality.
- Pairs $(N, \mathcal{F})$ with property 1 are *independence systems*.
- In fact, properties 1 and 2 are equivalent to our original definition and properties 2 and 3 are equivalent.
- The system defining the associated submodular polyhedron is TDI.
- Optimizing over the associated submodular polyhedron solves the maximum weight independent set problem.
- An independence system is a matroid if and only if the greedy algorithm solves the maximum weight independent set problems for every cost function.

Common Matroids

- Matric Matroids
  - Ground set is the set of columns/rows of a matrix.
  - Independent sets are the sets of linearly independent rows/columns.
- Graphic Matroid
  - The ground set is the set of edges of a graph.
  - Independent sets are the sets of edges of the graph that do not form a cycle.
- Partition Matroid
  - Ground set is the union of m finite disjoint sets E_i for $i = 1, \ldots, r$.
  - Independent sets are sets formed by taking at most one element from each set E_i.