Reading for This Lecture

- N&W Sections I.4.1-I.4.3
Some Conventions

If not otherwise stated, the following conventions will be followed for lecture slides during the course:

- $A$ will denote a matrix of dimension $m$ by $n$ (rational).
- $b$ will denote a vector of dimension $m$ (rational).
- $x$ will denote a vector of dimension $n$.
- $c$ will denote a vector of dimension $n$ (rational).
- $p$ will be the number of integer variables.
- $\mathcal{P}$ will denote a polyhedron contained in $\mathbb{R}^n$, usually given in the form
  \[ \mathcal{P} = \{ x \in \mathbb{R}^n \mid Ax \leq b \} \]
- $\mathcal{S}$ will be $\mathcal{P} \cap (\mathbb{Z}_+^p \times \mathbb{R}_+^{n-p})$.
- An integer program is then described fully by the quadruplet $(A, b, c, p)$.
- Vectors will be column vectors unless otherwise noted.
- When taking the product of vectors, we will sometimes leave off the transpose.
Additional Notation

• The notation $A_N$ will denote a submatrix formed by taking the columns indexed by set $N \subseteq \{1, \ldots, n\}$.

• The $i^{\text{th}}$ column of $A$ will be denoted $A_i$.

• The $i^{\text{th}}$ row of $A$ will be denoted $a_i$. 
Linear Algebra Review: Linear Independence

Definition 1. A finite collection of vectors $x^1, \ldots, x^k \in \mathbb{R}^n$ is linearly independent if the unique solution to $\sum_{i=1}^{k} \lambda_i x^i = 0$ is $\lambda_i = 0, i \in [1..k]$. Otherwise, the vectors are linearly dependent.

Let $A$ be a square matrix. Then, the following statements are equivalent:

- The matrix $A$ is invertible.
- The matrix $A^\top$ is invertible.
- The determinant of $A$ is nonzero.
- The rows of $A$ are linearly independent.
- The columns of $A$ are linearly independent.
- For every vector $b$, the system $Ax = b$ has a unique solution.
- There exists some vector $b$ for which the system $Ax = b$ has a unique solution.
Linear Algebra Review: Affine Independence

Definition 2. A finite collection of vectors $x^1, \ldots, x^k \in \mathbb{R}^n$ is affinely independent if the vectors $x^2 - x^1, \ldots, x^k - x^1 \in \mathbb{R}^n$ are linearly independent.

- Linear independence implies affine independence, but not vice versa.
- The property of linear independence is with respect to a given origin.
- Affine independence is essentially a “coordinate-free” version of linear independence.

Proposition 1. The following statements are equivalent:

1. $x_1, \ldots, x_k \in \mathbb{R}^n$ are affinely independent.
2. $x_2 - x_1, \ldots, x_k - x_1$ are linearly independent.
3. $(x_1, 1), \ldots, (x_k, 1) \in \mathbb{R}^{n+1}$ are linearly independent.
Linear Algebra Review: Subspaces

Definition 3. A nonempty subset $H \subseteq \mathbb{R}^n$ is called a subspace if $\alpha x + \gamma y \in H \ \forall x, y \in H$ and $\forall \alpha, \gamma \in \mathbb{R}$.

Definition 4. A linear combination of a collection of vectors $x^1, \ldots, x^k \in \mathbb{R}^n$ is any vector $y \in \mathbb{R}^n$ such that $y = \sum_{i=1}^{k} \lambda_i x^i$ for some $\lambda \in \mathbb{R}^k$.

Definition 5. The span of a collection of vectors $x^1, \ldots, x^k \in \mathbb{R}^n$ is the set of all linear combinations of those vectors.

Definition 6. Given a subspace $H \subseteq \mathbb{R}^n$, a collection of linearly independent vectors whose span is $H$ is called a basis of $H$. The number of vectors in the basis is the dimension of the subspace.
Linear Algebra Review: Subspaces and Bases

- A given subspace has an infinite number of bases.
- Each basis has the same number of vectors in it.
- If $S$ and $T$ are subspaces such that $S \subseteq T \subseteq \mathbb{R}^n$, then a basis of $S$ can be extended to a basis of $T$.
- The span of the columns of a matrix $A$ is a subspace called the column space or the range, denoted $\text{range}(A)$.
- The span of the rows of a matrix $A$ is a subspace called the row space.
- The dimensions of the column space and row space are always equal. We call this number $\text{rank}(A)$.
- Clearly, $\text{rank}(A) \leq \min\{m, n\}$. If $\text{rank}(A) = \min\{m, n\}$, then $A$ is said to have full rank.
- The set $\{x \in \mathbb{R}^n \mid Ax = 0\}$ is called the nullspace of $A$ (denoted $\text{null}(A)$) and has dimension $n - \text{rank}(A)$. 
Some Properties of Subspaces

Proposition 2. The following are equivalent:

1. \( H \subseteq \mathbb{R}^n \) is a subspace.
2. There is an \( m \times n \) matrix \( A \) such that \( H = \{ x \in \mathbb{R}^n \mid Ax = 0 \} \).
3. There is a \( k \times n \) matrix \( B \) such that \( H = \{ x \in \mathbb{R}^n \mid x = uB, u \in \mathbb{R}^k \} \).

Proposition 3. If \( \{ x \in \mathbb{R}^n \mid Ax = b \} \neq \emptyset \), the maximum number of affinely independent solutions of \( Ax = b \) is \( n + 1 - \text{rank}(A) \).

Proposition 4. If \( H \subseteq \mathbb{R}^n \) is a subspace, the subspace \( \{ x \in \mathbb{R}^n \mid x^\top y = 0 \forall y \in H \} \) is a subspace called the orthogonal subspace and denoted \( H^\perp \).

Proposition 5. If \( H = \{ x \in \mathbb{R}^n \mid Ax = 0 \} \), with \( A \) being an \( m \times n \) matrix, then \( H^\perp = \{ x \in \mathbb{R}^n \mid x = A^\top u, u \in \mathbb{R}^m \} \).
Affine Spaces

Definition 7. An affine combination of a collection of vectors $x^1, \ldots, x^k \in \mathbb{R}^n$ is any vector $y \in \mathbb{R}^n$ such that $y = \sum_{i=1}^{k} \lambda_i x^i$ for some $\lambda \in \mathbb{R}^k$ with $\sum_{j=1}^{k} \lambda_j = 1$.

Definition 8. A nonempty subset $A \subseteq \mathbb{R}^n$ is called an affine space if $A$ is closed with respect to affine combination.

Definition 9. A basis of an affine space $A \subseteq \mathbb{R}^n$ is maximal set of affinely independent points of $A$.

Definition 10. The inclusionwise minimal affine space containing a set $S$ is called the affine hull of $S$, denoted $\text{aff}(S)$.

Definition 11. All bases of an affine space $A$ have the same cardinality and this is the dimension of the affine space.


Projections

**Definition 12.** If \( p \in \mathbb{R}^n \) and \( H \) is a subspace, the projection of \( p \) onto \( H \) is the vector \( q \in H \) such that \( p - q \in H^\perp \).

- Note that this is a decomposition of a vector \( p \) into the sum of a vector in \( H \) and a vector in \( H^\perp \).
- The projection of a set is the union of the projections of all its members.
- Projections play a very important role in discrete optimization, as we will see later in the course.
Polyhedra, Hyperplanes, and Half-spaces

Definition 13. A polyhedron is a set of the form \( \{x \in \mathbb{R}^n \mid Ax \leq b\} \), where \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \).

Definition 14. A polyhedron \( \mathcal{P} \subset \mathbb{R}^n \) is bounded if there exists a constant \( K \) such that \( |x_i| < K \forall x \in S, \forall i \in [1, n] \).

Definition 15. A bounded polyhedron is called a polytope.

Definition 16. Let \( a \in \mathbb{R}^n \) and \( b \in \mathbb{R} \) be given.

- The set \( \{x \in \mathbb{R}^n \mid a^\top x = b\} \) is called a hyperplane.
- The set \( \{x \in \mathbb{R}^n \mid a^\top x \leq b\} \) is called a half-space.
Convex Sets

Definition 17. A set $S \subseteq \mathbb{R}^n$ is convex if $\forall x, y \in S, \lambda \in [0, 1]$, we have $\lambda x + (1 - \lambda)y \in S$.

Definition 18. Let $x^1, \ldots, x^k \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^k_+$ be given such that $\lambda^\top \mathbf{1} = 1$. Then

1. The vector $\sum_{i=1}^{k} \lambda_i x^i$ is said to be a convex combination of $x^1, \ldots, x^k$.

2. The convex hull of $x^1, \ldots, x^k$ is the set of all convex combinations of these vectors.

- The convex hull of two points is a line segment.
- A set is convex if and only if for any two points in the set, the line segment joining those two points lies entirely in the set.
- All polyhedra are convex.