Reading for This Lecture

- Wolsey Section 2.5
- Nemhauser and Wolsey II.3.1-II.3.3
- “Duality for Mixed-Integer Linear Programs,” Guzelsoy and
Duality

- An alternative to relaxation for obtaining bounds is to formulate a dual problem.

- Let a pure integer program be defined by

\[ z_{IP} = \max \{ cx \mid x \in S \}, \quad S = \{ x \in \mathbb{Z}_+^n \mid Ax = b \} \]

where \( c \in \mathbb{R}^n, A \in \mathbb{Q}^{m \times n}, b \in \mathbb{R}^m \).

- We refer to this instance as the primal problem.

- A weak dual problem is an optimization problem of the form

\[ z_D = \min_{v \in V} f(v), \]

with \( f : V \to \mathbb{R}, V \subseteq \mathbb{R}^k \) for \( k \in \mathbb{N} \) such that \( z_D \geq z_{IP} \).

- A strong dual is a weak dual if \( z_{IP} \) is finite and also \( z_D = z_{IP} \).
Importance of Duality

- Note again that if we have a dual to $IP$, then we can easily obtain bounds on the value of an optimal solution.
- The advantage of a dual is that we need not solve it to optimality.
- Any feasible solution to the dual yields a valid bound.
- The three main categories of duals used most frequently are:
  - LP duals
  - Combinatorial duals
  - Lagrangian duals
The Duality Gap

- In the case of weak duals, there is a gap between the optimal solution to the dual problem and the optimal solution to IP.
- This gap is known as the duality gap or just the gap.
- It is typically measured as a percentage of the value of an optimal solution.
- The size of the gap is a measure of the difficulty of a problem.
- It can help us estimate how long it will take to solve a given problem by branch and bound.
- As a rule of thumb, problems with a gap of more than 5-10% are too difficult to solve in practice.
- Note that in most cases, we don’t know the exact gap because we don’t know the exact value of an optimal solution.
- Usually, the gap is estimated based on the best known solution.
Generalized Dual

- The previously introduced definition of a dual problem is not useful, since the dual problem is not selected by any measure of goodness.

- Conceptually, we can improve the situation by choosing the "best" from a family of dual problems to obtain

\[ z_D = \min_{f,V} \min_{v \in V} f(v), \]

where each pair \((f, V)\) is required to comprise a dual problem.

- It is not clear how to solve this generalized dual.

- Even if this were possible, the dual would not be useful for analyzing perturbed instances, as the LP dual is.

- We would like to generalize the concept of duality we had in LP in order to be able to perform sensitivity analyses and warm start.

- What we would like is a *dual function* that can produce a valid bound across a range of perturbed instances.

- This is essentially what we have in the LP case.
Value function

- What do we mean by the *neighborhood* of a given instance?
- Here, we only consider varying the right-hand-side.
- The *value function* is defined as:

\[
z(d) = \max_{x \in S(d)} cx,
\]

where \( S(d) = \{x \in \mathbb{Z}_+^n \mid Ax = d\} \), \( d \in \mathbb{R}^m \).
- We let \( z(d) = -\infty \) if \( d \notin \Omega \), where \( \Omega = \{d \in \mathbb{R}^m \mid S(d) \neq \emptyset \} \).
Example

Consider the following instance

$$z_{IP} = \max \quad -\frac{1}{2} x_1 - 2x_3 - x_4$$
$$\text{s.t} \quad x_1 - \frac{3}{2}x_2 + x_3 - x_4 = b \quad \text{and}$$
$$x_i \in \mathbb{Z}_+, i = 1, \ldots, 4$$

In closed form, we have $$z(d) = -\frac{3}{2} \max\{\lceil \frac{2d}{3} \rceil, \ lceil d \rceil\} + d, d \in \Omega$$

See Figure.
**Dual Function**

- It is difficult to construct the value function itself.
- It is easier to obtain an approximate function that bounds the value function from above.
  - A *dual function* $F : \mathbb{R}^m \to \mathbb{R}$ is one that satisfies $F(d) \geq z(d)$ for all $d \in \mathbb{R}^m$.
  - What is a “good” dual function?
  - We can choose one that provides the best bound for the current right-hand side $b$.
  - This results in the dual

\[
z_D = \min \{ F(b) : F(d) \geq z(d), \; d \in \mathbb{R}^m, \; F \in \mathcal{Y}^m \}
\]

where $\mathcal{Y}^m \subseteq \{ f : \mathbb{R}^m \to \mathbb{R} \}$.

- We call $F^*$ *strong* if $F^*$ is a *feasible* dual function and $F^*(b) = z_{IP}$.
- This dual problem always has a solution $F^*$ that is strong if the primal problem is finite and $\mathcal{Y}^m \equiv \{ f : \mathbb{R}^m \to \mathbb{R} \}$. Why?
A Dual Function from LP Relaxation

- Consider the value function of the LP relaxation of the primal problem:

$$F_{LP}(d) = \min \{vd : vA \geq c, \ v \in \mathbb{R}^m\}.$$ 

- By linear programming duality theory, we have $F_{LP}(d) \geq z(d)$ for all $d \in \mathbb{R}^m$.

- $F_{LP}$ is not necessarily strong.
Example

Consider the value function of the LP relaxation of our instance

\[ F_{LP}(d) = \min \ v d, \]
\[ \text{s.t} \quad 0 \geq v \geq -\frac{1}{2}, \text{ and} \]
\[ v \in \mathbb{R}, \]

which can be written explicitly as

\[ F_{LP}(d) = \begin{cases} 
0, & d \leq 0 \\
-\frac{1}{2}d, & d > 0 
\end{cases}. \]

See Figure.
**The Superadditive Dual**

By considering that

\[ F(d) \geq z(d), \ d \in \mathbb{R}^m \quad \iff \quad F(d) \geq cx, \ x \in S(d), \ d \in \mathbb{R}^m \]
\[ \iff \quad F(Ax) \geq cx, \ x \in \mathbb{Z}_+^n, \]

the dual problem can be rewritten as

\[ z_D = \min \ \{ F(b) : F(Ax) \geq cx, \ x \in \mathbb{Z}_+^n, \ F \in \mathcal{Y}^m \} \]

Can we further restrict \( \mathcal{Y}^m \) and still guarantee a strong dual solution?

- The class of linear functions? NO!
- The class of concave functions? NO!
- The class of superadditive functions? YES!
The Superadditive Dual

- Let a function $F$ be defined over a domain $V$. Then $F$ is superadditive if $F(v_1) + F(v_2) \leq F(v_1 + v_2) \forall v_1, v_2, v_1 + v_2 \in V$.

- A strong motivation: value function $z$ is superadditive over $\Omega$.

If $\mathcal{V}^m \equiv \Gamma^m \equiv \{F \text{ is superadditive} \mid F : \mathbb{R}^m \rightarrow \mathbb{R}, F(0) = 0\}$, we can rewrite the dual problem above as the superadditive dual

$$Z_D = \min \ F(b)$$

$$F(a^j) \geq c_j \quad j = 1, ..., n,$$

$$F \in \Gamma^m$$

where $a^j$ is the $j^{th}$ column of $A$. 
Weak Duality

**Theorem 1.** Let $x$ be a feasible solution to the primal problem and let $F$ be a feasible solution to the superadditive dual. Then, $F(b) \geq cx$.

**Proof.**

**Corollary 1.** For the primal problem and its superadditive dual:

1. If the primal problem (resp., the dual) is unbounded then the dual problem (resp., the primal) is infeasible.

2. If the primal problem (resp., the dual) is infeasible, then the dual problem (resp., the primal) is infeasible or unbounded.
Strong Duality

**Theorem 2.** If the primal problem (resp., the dual) has a finite optimum, then so does the superadditive dual problem (resp., the primal) and they are equal.

**Outline of the Proof.** Show that the value function $z$ or an extension to $z$ is a feasible dual function.

- Note that $z$ satisfies the dual constraints.
- $\Omega \equiv \mathbb{R}^m$: $z \in \Gamma^m$.
- $\Omega \subset \mathbb{R}^m$: $\exists z_e \in \Gamma^m$ with $z_e(d) = z(d) \ \forall \ d \in \Omega$ and $z_e(d) < 1$.
Example

For our IP instance, the superadditive dual problem is

\[
\begin{align*}
\min & \quad F(b) \\
F(1) & \geq -\frac{1}{2} \\
F\left(-\frac{3}{2}\right) & \geq 0 \\
F(1) & \geq -2 \\
F(-1) & \geq -1 \\
F & \in \Gamma^1.
\end{align*}
\]
Example

Notice how optimal solutions for different right-hand-sides give different bounds for other right-hand-sides.

1. $F_1(d) = -\frac{d}{2}$ is an optimal dual function for $b \in \{0, 1, 2, \ldots\}$.

2. $F_2(d) = 0$ is an optimal function for $b \in \{-3, -\frac{3}{2}, 0\}$.

3. $F_3(d) = -\max\left\{\frac{1}{2}\left[d - \left\lfloor\left\lfloor\frac{d}{4}\right\rfloor - d\right\rfloor\right], 2d - \frac{3}{2}\left[d - \left\lfloor\left\lfloor\frac{d}{4}\right\rfloor - d\right\rfloor\right]\right\}$ is an optimal function for $b \in \{[0, \frac{1}{4}] \cup [1, \frac{5}{4}] \cup [2, \frac{9}{4}] \cup \ldots\}$.

4. $F_4(d) = -\max\left\{\frac{3}{2}\left[\frac{2d}{3} - \frac{2}{3}\left\lfloor\left\lfloor\frac{2d}{3}\right\rfloor - \frac{2d}{3}\right\rfloor\right] - d, -\frac{3}{4}\left[\frac{2d}{3} - \frac{2}{3}\left\lfloor\left\lfloor\frac{2d}{3}\right\rfloor - \frac{2d}{3}\right\rfloor\right]\right\}$ is an optimal function for $b \in \{\ldots \cup [-\frac{7}{2}, -3] \cup [-2, -\frac{3}{2}] \cup [-\frac{1}{2}, 0]\}$.

See Figure.
\[ G(d) = \min \{ F_1(d), F_2(d) \} \]
\[ H(d) = \min \{ F_3(d), F_4(d) \} \]
Farkas’ Lemma

For the primal problem, exactly one of the following holds:

1. $S \neq \emptyset$

2. There is an $F \in \Gamma^m$ with $F(a^j) \geq 0, j = 1, ..., n$, and $F(b) < 0$.

Proof. Let $c = 0$ and apply strong duality theorem to superadditive dual.
Complementary Slackness

For a given right-hand side $b$, let $x^*$ and $F^*$ be feasible solutions to the primal and the superadditive dual problems, respectively. $x^*$ are optimal solutions if and only if

1. $x^*_j(c_j - F^*(a^j)) = 0, j = 1, \ldots, n$ and

2. $F^*(b) = \sum_{j=1}^{n} F^*(a^j)x^*_j$.

Proof. For an optimal pair we have

$$F^*(b) = F^*(Ax^*) = \sum_{j=1}^{n} F^*(a^j)x^*_j = cx^*.$$
Constructing Dual Functions

• Explicit construction
  – The Value Function
  – Generating Functions

• Relaxations
  – Lagrangian Relaxation
  – Quadratic Lagrangian Relaxation
  – Corrected Linear Dual Functions

• Primal Solution Algorithms
  – Cutting Plane Method
  – Branch-and-Bound Method
  – Branch-and-Cut Method
Branch-and-Bound Method

- Assume that the primal problem is solved to optimality. Let \( T \) be the set of leaf nodes.

- Note that we solve the LP relaxation of the following problem at node \( t \in T \):

\[
    z^t(b) = \max \ cx \\
    \text{s.t} \quad x \in S_t(b) ,
\]

where \( S_t(b) = \{ Ax = b, x \geq l^t, -x \geq -u^t, x \in \mathbb{Z}^n \} \) and \( u^t \) are the branching bounds applied to the integer variables.

- Let \((v^t, \underline{v}^t, \overline{v}^t)\) be

  - the dual feasible solution used to prune node \( t \), if \( t \) is feasibly pruned,
  - a dual feasible solution (that can be obtained from it parent) to node \( t \), if \( t \) is infeasibly pruned.
Dual Function from Branch-and-Bound Tree

Then,

\[ F_{BC}(d) = \max_{t \in T} \{ v^t d + \underline{v}^t l^t - \overline{v}^t u^t \} \]

is an optimal solution to the generalized dual problem.

Proof.

- We can also make use of the internal nodes.
- We can also get a dual function feasible to superadditive dual.
Superadditive Dual for Mixed-Integer Case

Let an MIP problem be defined by

\[ z_{IP} = \max \{ cx \mid x \in S \}, \quad S = \{ x \in \mathbb{Z}_+^r \times \mathbb{R}^{n-r}_+ \mid Ax = b \} \]

Then, its superadditive dual problem is

\[ z_D = \min \quad F(b) \]
\[ F(a^j) \geq c_j \quad j = 1, \ldots, r, \]
\[ \tilde{F}(a^j) \geq c_j \quad j = r + 1, \ldots, n, \quad \text{and} \]
\[ F \in \Gamma^m, \]

where the function \( \tilde{F} \) is defined by

\[ \tilde{F}(d) = \lim_{\delta \to 0^+} \sup \frac{F(\delta d)}{\delta} \quad \forall d \in \mathbb{R}^m. \]

Here, \( \tilde{F} \) is the upper \( d \)-directional derivative of \( F \) at zero.