

Integer Programming

ISE 418

Lecture 13a

Dr. Ted Ralphs

Reading for This Lecture

- Nemhauser and Wolsey Sections II.1.1-II.1.3, II.1.6
- Wolsey Chapter 8
- CCZ Chapters 5 and 6
- “Valid Inequalities for Mixed Integer Linear Programs,” G. Cornuejols.
- “Corner Polyhedra and Intersection Cuts,” M. Conforti, G. Cornuejols, and G. Zambelli.
- “Generating Disjunctive Cuts for Mixed Integer Programs,” M. Perregaard.

Valid Inequalities from Disjunctions

- Valid inequalities for $\text{conv}(\mathcal{S})$ can also be generated from valid disjunctions.
- Let $\{X_i\}_{i=1}^k$ constitute an admissible disjunction for \mathcal{S} .
- Then inequalities valid for $\cup_{i=1}^k (\mathcal{P} \cap X_i)$ are also valid for $\text{conv}(\mathcal{S})$.

The Union of Polyhedra

- The convex hull of the union of polyhedra is not necessarily a polyhedron.
- Under mild conditions, we can characterize it, however.
- Consider a finite collection of polyhedra $\mathcal{P}_i = \{x \in \mathbb{R}^n \mid A^i x \leq b^i\}$ for $1 \leq i \leq k$.
- Typically, we have $\mathcal{P}_i = X_i \cap \mathcal{P}$.
- Let Y be the polyhedron described by the following constraints:

$$\begin{aligned}
 A^i x^i &\leq b^i y_i \quad \forall i = 1, \dots, k \\
 \sum_{i=1}^k x^i &= x \\
 \sum_{i=1}^k y^i &= 1 \\
 y &\geq 0
 \end{aligned}$$

- Furthermore, for polyhedron \mathcal{P}_i , let $C_i = \{x \in \mathbb{R}^n \mid A^i x \leq 0\}$ and let $\mathcal{P}_i = Q_i + C_i$ where Q_i is a polytope.

The Convex Hull of the Union of Polyhedra

- Under the assumptions on the previous slide, we have the following result.

Theorem 1. *If either $\bigcup_{i=1}^k \mathcal{P}_i = \emptyset$ or $C_j \subseteq \text{cone}(\bigcup_{i:\mathcal{P}_i \neq \emptyset} C_i)$ for all j such that $\mathcal{P}_j = \emptyset$, then the following sets are identical:*

- $\overline{\text{conv}}(\bigcup_{i=1}^k \mathcal{P}_i)$
- $\text{conv}(\bigcup_{i=1}^k Q_i) + \text{cone}(\bigcup_{i=1}^k C_i)$
- $\text{proj}_x Y$.

- Note that the assumptions of the proposition are necessary, but are automatically satisfied if
 - $C^i = \{0\}$ whenever $\mathcal{P}^i = \emptyset$, or
 - all the polyhedra have the same recession cone.

Necessity of Assumption

- Let $\mathcal{P}_1 = \{x \in \mathbb{R}^2 \mid 0 \leq x \leq 1\}$ and $\mathcal{P}_2 = \{x \in \mathbb{R}^2 \mid x_1 \leq 0, x_1 \geq 1\}$.
- Then $\mathcal{P}_2 = \emptyset$, but $C_2 = \{\lambda(0, 1) \mid \lambda \in \mathbb{R}\}$.
- In this case, $\text{proj}_x Y = \mathcal{P}_1 + C_2$, which is not equal to $\text{conv}(\mathcal{P}_1 \cup \mathcal{P}_2) = \mathcal{P}_1$.

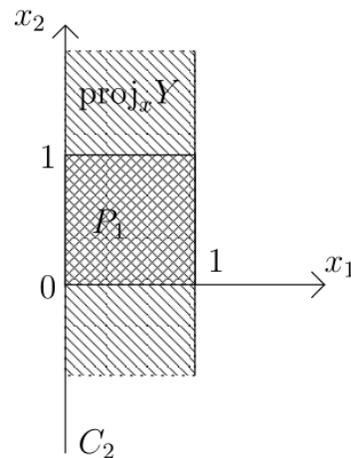


Figure 1: Example showing necessity of assumptions in Theorem 1

The Convex Hull of the Union of Polyhedra (cont.)

- Note also that if all the polyhedra have the same recession cones, then $\overline{\text{conv}}(\cup_{i=1}^k \mathcal{P}_i) = \text{conv}(\cup_{i=1}^k \mathcal{P}_i)$ and $\cup_{i=1}^k \mathcal{P}_i$ is the projection of

$$\begin{aligned} A^i x^i &\leq b^i y_i \quad \forall i = 1, \dots, k \\ \sum_{i=1}^k x^i &= x \\ \sum_{i=1}^k y_i &= 1 \\ y &\in \{0, 1\} \end{aligned}$$

- This is the case when the polyhedra only differ in their right-hand sides, as is the case when branching on variables.
- This is the formulation used to prove one direction of the Theorem on Mixed Integer Representability.

Valid Inequalities from Disjunctions

Another viewpoint for constructing valid inequalities based on disjunctions comes from the following result:

Proposition 1. *If (π^1, π_0^1) is valid for $\mathcal{S}_1 \subseteq \mathbb{R}_+^n$ and (π^2, π_0^2) is valid for $\mathcal{S}_2 \subseteq \mathbb{R}_+^n$, then*

$$\sum_{j=1}^n \min(\pi_j^1, \pi_j^2) x_j \leq \max(\pi_0^1, \pi_0^2) \quad (1)$$

for $x \in \mathcal{S}_1 \cup \mathcal{S}_2$.

In fact, all valid inequalities for the union of two polyhedra can be obtained in this way.

Proposition 2. *If $\mathcal{P}^i = \{x \in \mathbb{R}_+^n \mid A^i x \leq b^i\}$ for $i = 1, 2$ are nonempty polyhedra, then (π, π_0) is a valid inequality for $\text{conv}(\mathcal{P}^1 \cup \mathcal{P}^2)$ if and only if there exist $u^1, u^2 \in \mathbb{R}^m$ such $\pi \leq u^i A^i$ and $\pi_0 \geq u^i b^i$ for $i = 1, 2$.*

Simple Disjunctive Inequalities

- We want to develop a procedure analogous to C-G for mixed-integer sets.
- It is straightforward to develop an analog of the rounding principle we used earlier that was geared towards pure integer programs.

Proposition 3. Let $T = \{x \in \mathbb{Z} \times \mathbb{R}_+ \mid x_1 - x_2 \leq b\}$. Then the inequality

$$x_1 - \frac{1}{1 - f_0} x_2 \leq \lfloor b \rfloor.$$

is valid for T , where $f_0 := b - \lfloor b \rfloor$.

- The proof requires exploiting the disjunction

$$x_1 \leq \lfloor b \rfloor \text{ OR } x_1 \geq \lfloor b \rfloor + 1$$

Proof of Validity

There are two cases.

- If $x \leq \lfloor b \rfloor$, then adding this to $\frac{1}{1-f_0}$ times $x_2 \leq 0$ yields (3).
- If $x \geq \lfloor b \rfloor + 1$, then adding $\frac{f_0}{1-f_0}$ times $-x_1 \leq -\lfloor b \rfloor - 1$ to $\frac{1}{1-f_0}$ times $x_1 - x_2 \leq b$ yields (3).

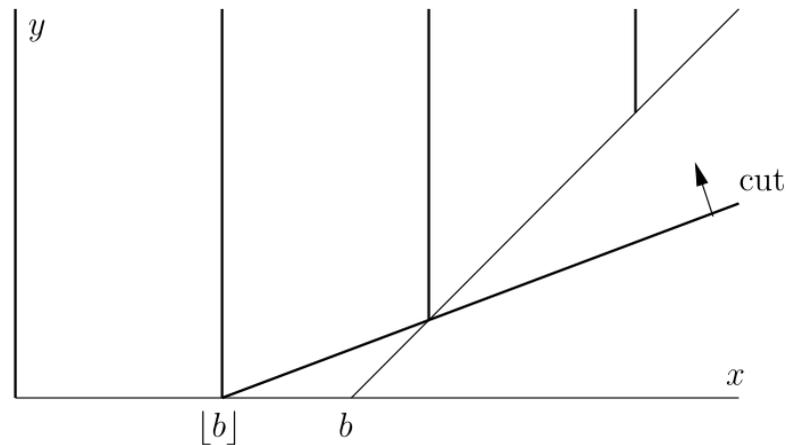


Figure 2: Figure illustrating the principle from CCZ

Mixed Integer Rounding Inequalities

- We can generalize the inequality from the previous slide by aggregating variables.
- Let $\alpha \in \mathbb{R}^n$ be an integral vector and let (π, π_0) be an inequality valid for \mathcal{P} .
- Then $\alpha x \in \mathbb{Z}$ and $\pi_0 - \pi x \geq 0$ for all $x \in \mathcal{S}$.
- Therefore, if we have

$$\alpha x - (\pi_0 - \pi x) \leq \beta \quad \forall x \in \mathcal{P}$$

we can apply the simple disjunctive procedure from the last slide to obtain that

$$\alpha x - \frac{1}{1-f}(\pi_0 - \pi x) \leq \lfloor \beta \rfloor \quad \forall x \in \mathcal{P}$$

Another Variant

- If we further assume non-negativity of the variables, we can derive a variant equivalent to the GMIC.

Proposition 4. Let $T = \{x \in \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p} \mid \sum_{j=1}^n a_j x_j \leq b\}$, where $a \in \mathbb{Q}^n$ and $b \in \mathbb{Q}$. Then the inequality

$$\sum_{j=1}^p (\lfloor a_j \rfloor + \frac{(f_j - f_0)^+}{1 - f_0}) x_j + \frac{1}{1 - f_0} \sum_{p+1 \leq j \leq n: a_j < 0} a_j x_j \leq \lfloor b \rfloor. \quad (\text{MIR})$$

is valid for T , where $f_j = a_j - \lfloor a_j \rfloor$ and $f_0 = b - \lfloor b \rfloor$.

- In fact, if $a_j \in \mathbb{Z}$, $\gcd\{a_1, \dots, a_n\} = 1$, and $b \notin \mathbb{Z}$, then the above inequality is facet-inducing for $\text{conv}(T)$.
- The above inequality is called a *mixed integer rounding* (MIR) inequality.
- Its validity can be proved by aggregating the integer and continuous variables, respectively, and applying Proposition 3.

Proof of Validity

- Because we assume the variables are non-negative, we can relax the inequality to

$$\sum_{\substack{0 \leq j \leq p \\ f_j \leq f_0}} \lfloor a_j \rfloor x_j + \sum_{\substack{0 \leq j \leq p \\ f_j > f_0}} a_j x_j + \sum_{\substack{p+1 \leq j \leq n \\ a_j < 0}} a_j x_j \leq b$$

- Then let

$$w := \sum_{\substack{0 \leq j \leq p \\ f_j \leq f_0}} \lfloor a_j \rfloor x_j + \sum_{\substack{0 \leq j \leq p \\ f_j > f_0}} \lceil a_j \rceil x_j$$

$$z := - \sum_{\substack{p+1 \leq j \leq n \\ a_j < 0}} a_j x_j + \sum_{\substack{0 \leq j \leq p \\ f_j > f_0}} (1 - f_j) x_j$$

- Then we have $(w - z, b)$ is valid for T , $w \in \mathbb{Z}$, and $z \in \mathbb{R}_+$, so

$$w - \frac{1}{1 - f_0} z \leq \lfloor b \rfloor$$

which yields that (MIR) is valid for T .

Gomory Mixed Integer Inequalities

- Let's consider again the set of solutions T to an IP with one equation.
- This time, we write T equivalently as

$$T = \left\{ x \in \mathbb{Z}_+^n \mid \sum_{j:f_j \leq f_0} f_j x_j + \sum_{j:f_j > f_0} (f_j - 1)x_j = f_0 - k \right\},$$

where $k = \lfloor a_0 \rfloor + \sum_{\substack{0 \leq j \leq p \\ f_j \leq f_0}} \lfloor a_j \rfloor x_j + \sum_{\substack{0 \leq j \leq p \\ f_j > f_0}} \lceil a_j \rceil x_j$.

- Since $k \in \mathbb{Z}$, the disjunction $k \leq -1$ OR $k \geq 0$ is valid so we have

$$\sum_{j:f_j \leq f_0} \frac{f_j}{f_0} x_j - \sum_{j:f_j > f_0} \frac{(1 - f_j)}{f_0} x_j \geq 1$$

OR

$$- \sum_{j:f_j \leq f_0} \frac{f_j}{(1 - f_0)} x_j + \sum_{j:f_j > f_0} \frac{(1 - f_j)}{(1 - f_0)} x_j \geq 1$$

The Gomory Mixed Integer Cut

- Applying Proposition 1, we get

$$\sum_{j:f_j \leq f_0} \frac{f_j}{f_0} x_j + \sum_{j:f_j > f_0} \frac{(1-f_j)}{(1-f_0)} x_j \geq 1$$

- This is called a *Gomory mixed integer* (GMI) inequality.
- GMI inequalities dominate the associated Gomory cut and can also be obtained easily from the tableau.
- We can easily generalize to the mixed integer set

$$T = \left\{ x \in \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p} \mid \sum_{j=1}^n a_j x_j = a_0 \right\},$$

to obtain the GMI cut

$$\sum_{\substack{0 \leq j \leq p \\ f_j \leq f_0}} \frac{f_j}{f_0} x_j + \sum_{\substack{0 \leq j \leq p \\ f_j > f_0}} \frac{(1-f_j)}{(1-f_0)} x_j + \sum_{\substack{p+1 \leq j \leq n \\ a_j > 0}} \frac{a_j}{f_0} x_j - \sum_{\substack{p+1 \leq j \leq n \\ a_j < 0}} \frac{a_j}{(1-f_0)} x_j \geq 1$$

(GMIC)

GMI vs. MIR

- Although we derived the GMI inequality using a different logic than that which we used for the MIR inequality, they are equivalent.
- Beginning with the inequality

$$\sum_{j=1}^n a_j x_j \leq b, \quad (2)$$

we add a slack variables $s = b - a^\top x$ to obtain

$$\sum_{j=1}^n a_j x_j + s = b.$$

- Deriving the GMI inequality from this equation and then substituting out the slack variable s , we obtain the MIR inequality associated with (2).
- Note that both inequalities are derived from the same split disjunction.
- We will see later that all inequalities derived from split disjunctions are of a form equivalent to (GMIC) or (MIR).

Gomory Mixed Integer Cuts from the Tableau

- Let's consider how to generate Gomory mixed integer cuts from the tableau.
- As before, we first introduce a slack variable for each inequality in the formulation.
- Solving the LP relaxation, we look for a row in the tableau in which an integer variable is basic and has a fractional variable.
- We apply the GMI procedure to produce a cut.
- Finally, we substitute out the slack variables in order to express the cut in terms of the original variables only.

Example: GMI Cuts versus Gomory Cuts

Recall our example from last time.

$$\max \quad 2x_1 + 5x_2 \quad (3)$$

$$\text{s.t.} \quad 4x_1 + x_2 \leq 28 \quad (4)$$

$$x_1 + 4x_2 \leq 27 \quad (5)$$

$$x_1 - x_2 \leq 1 \quad (6)$$

$$x_1, x_2 \geq 0 \quad (7)$$

The optimal tableau for the LP relaxation is:

Basic var.	x_1	x_2	s_1	s_2	s_3	RHS
x_2	0	1	-2/30	8/30	0	16/3
s_3	0	0	-1/3	1/3	1	2/3
x_1	1	0	8/30	-2/30	0	17/3

Table 1: Optimal tableau of the LP relaxation

The associated optimal solution to the LP relaxation is also shown in Figure 3.

Example: Gomory Cuts (cont.)

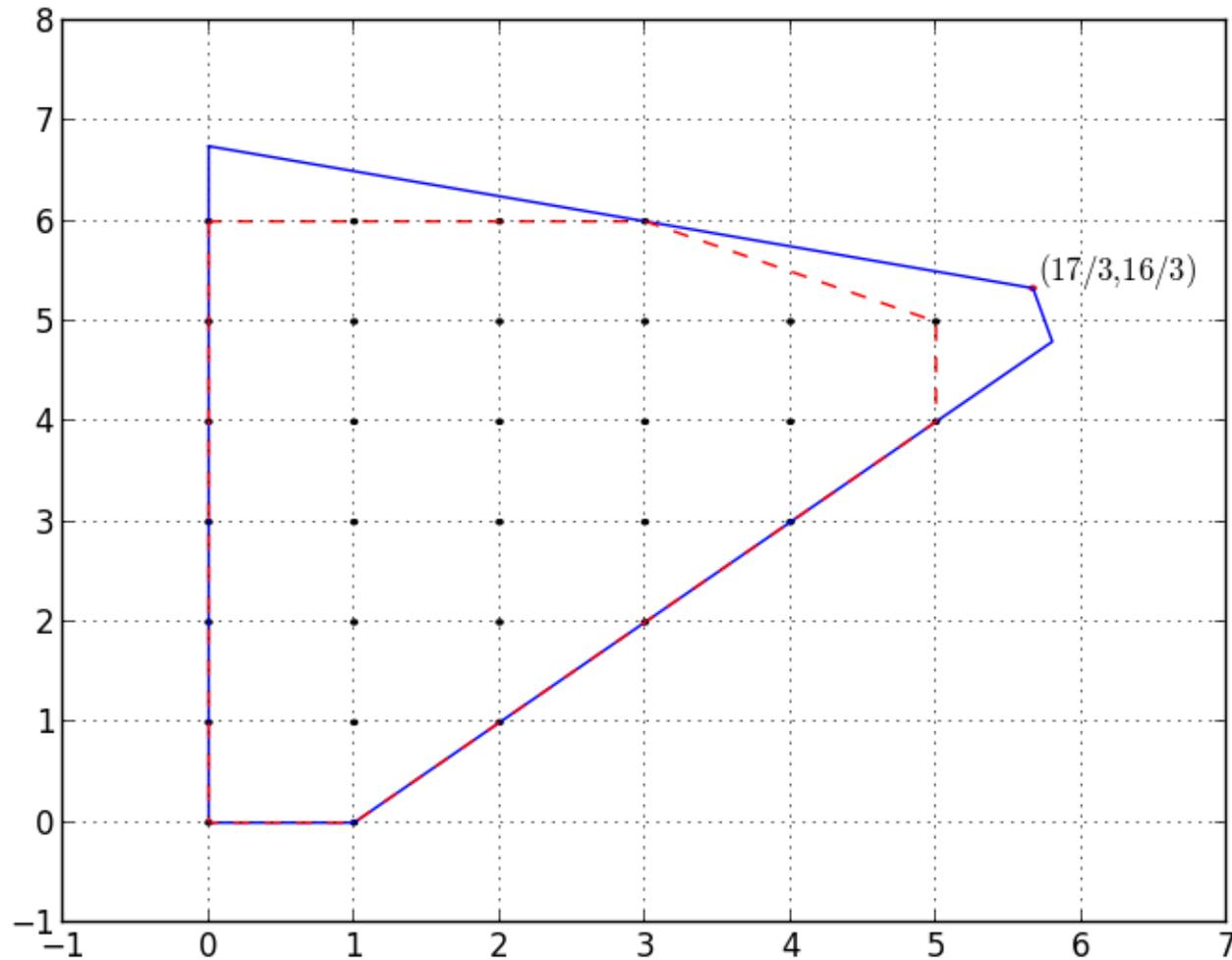


Figure 3: Convex hull of S

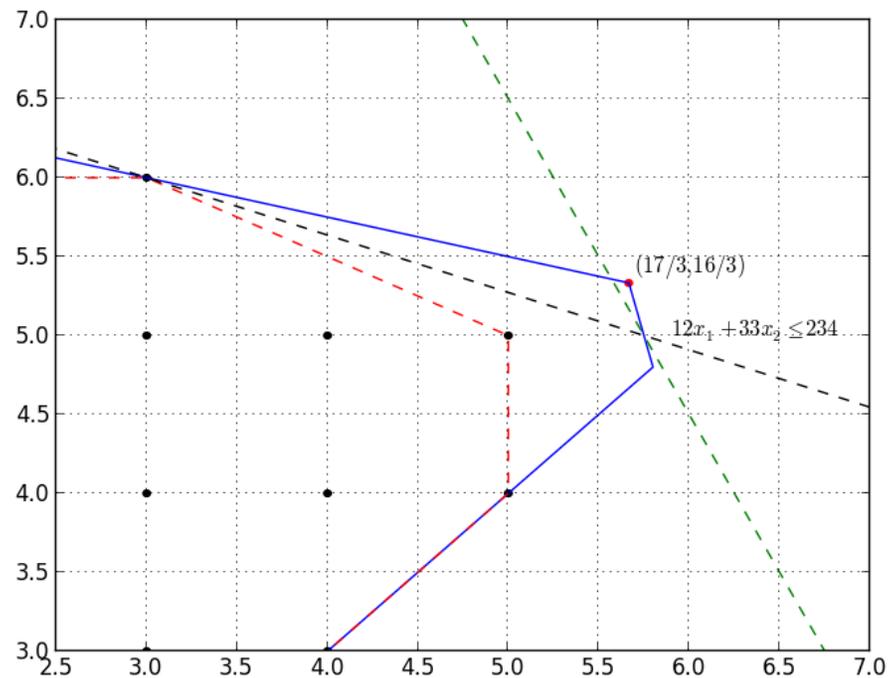
Example: GMI Cuts versus Gomory Cuts (cont.)

The GMI cut from the first row is

$$\frac{1}{10}s_1 + \frac{8}{10}s_2 \geq 1,$$

In terms of x_1 and x_2 , we have

$$12x_1 + 33x_2 \leq 234, \quad (\text{GMI-C1})$$



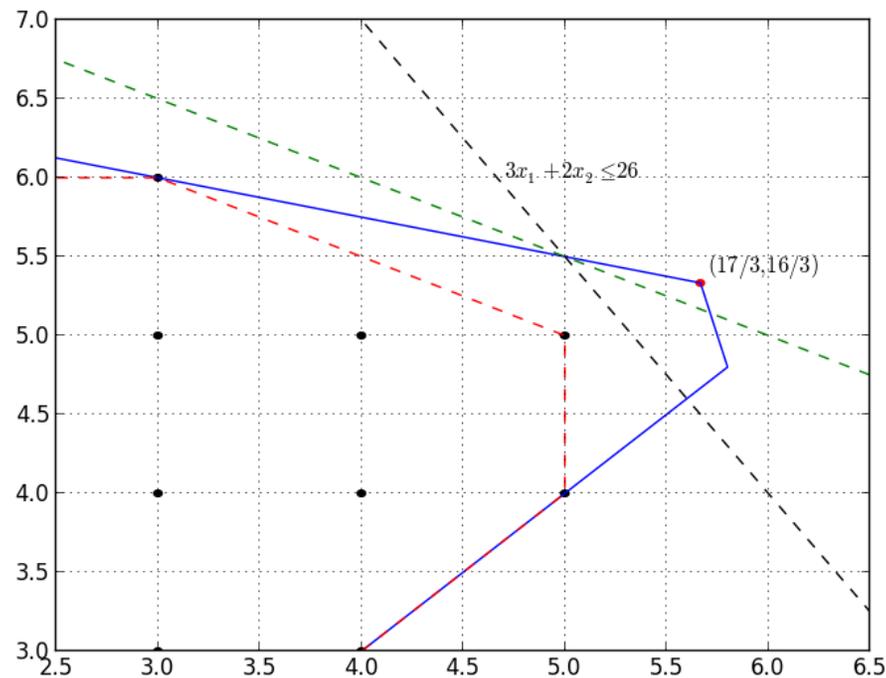
Example: GMI Cuts versus Gomory Cuts (cont.)

The GMI cut from the third row is

$$\frac{4}{10}s_1 + \frac{2}{10}s_2 \geq 1,$$

In terms of x_1 and x_2 , we have

$$3x_1 + 2x_2 \leq 26, \quad (\text{GMI-C3})$$



Geometric Interpretation of GMI Cuts

- To understand the geometric interpretation of GMI cuts, we consider a relaxation associated with a basis of the LP relaxation.
- We simply relax the non-negativity constraints on the basic variables to obtain

$$T = \{ (x, s) \in \mathbb{Z}^{n+m} \mid Ax + Is = b, x_N \geq 0, s_N \geq 0 \},$$

where x_N and s_N are the non-basic variables associated with basis B .

- This is equivalent to relaxing the non-binding constraints.
- The convex hull of T is the so-called *corner polyhedron* associated with the basis B .

Example: Corner Polyhedron

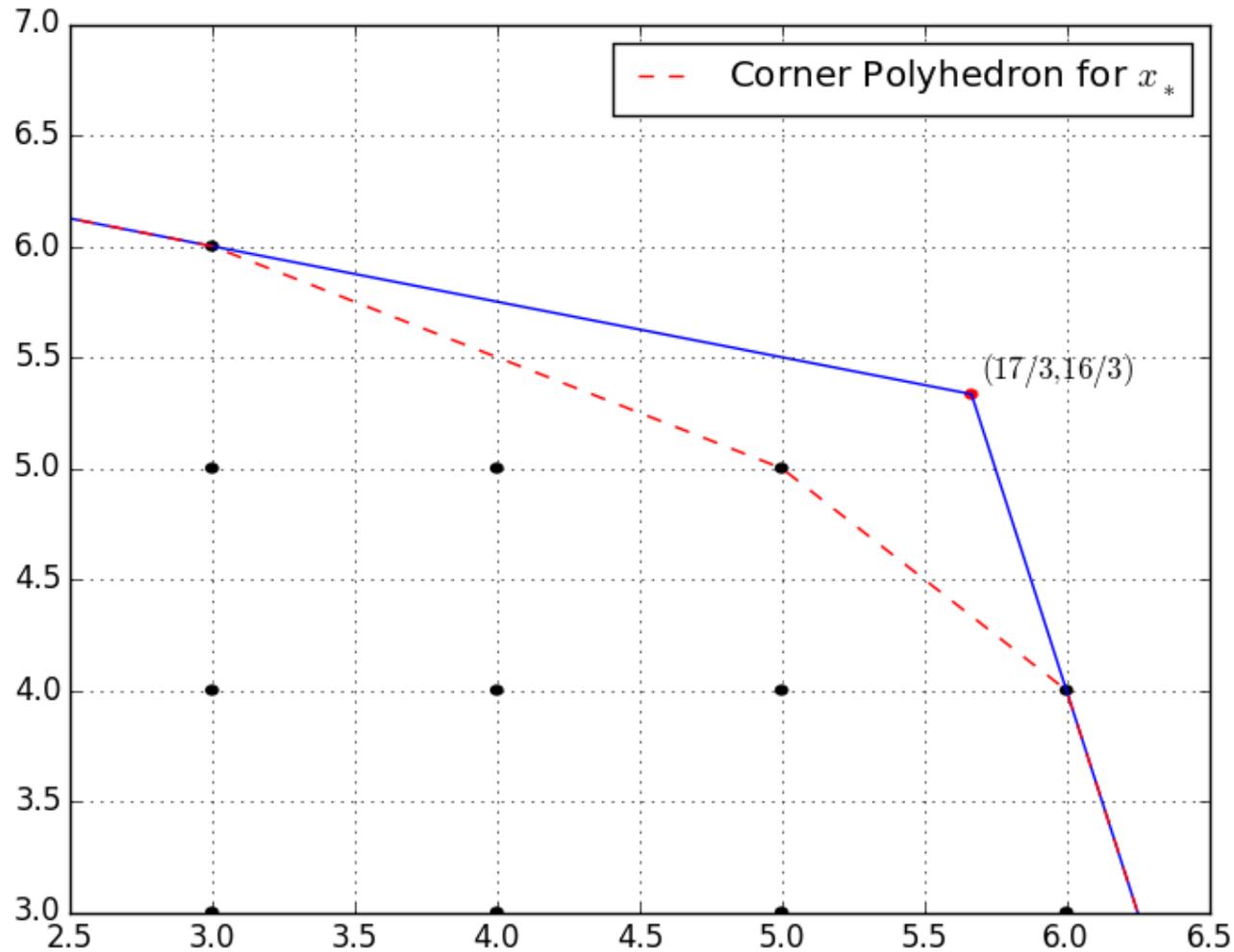


Figure 4: The corner polyhedron associated with the optimal basis of the LP relaxation of the earlier example.

Intersection Cuts

- A simple way to obtain inequalities valid for the corner polyhedron (and hence the original MILP) is as follows.
 - Construct a convex set C whose interior contains the solution x^* associated with the current basis and no other integer points.
 - Determine the points of intersection of each of the extreme rays of the corner polyhedron with the set C .
 - The unique hyperplane determined by these points of intersection then separates x^* from the corner polyhedron.
- This is a general paradigm and one can get different classes of valid inequality by choosing the set C in different ways.
- The GMI cut from row i of the tableau is precisely the intersection cut obtained by setting

$$C = \{x \in \mathbb{R}^{n+m} \mid \lfloor x_j^* \rfloor \leq x_j \leq \lceil x_j^* \rceil\},$$

where x_j is the variable that is basic in row i .

Example: GMI Cut as an Intersection Cut

- Figure 5 shows the GMI cut derived from the second row of the tableau in our example as an intersection cut.
- The basic variable in this case is s_3 .
- In terms of the original variables, we have

$$C = \{x \in \mathbb{R}^{n+m} \mid 0 \leq x_1 - x_2 \leq 1\}$$

Example: GMI Cut as an Intersection Cut

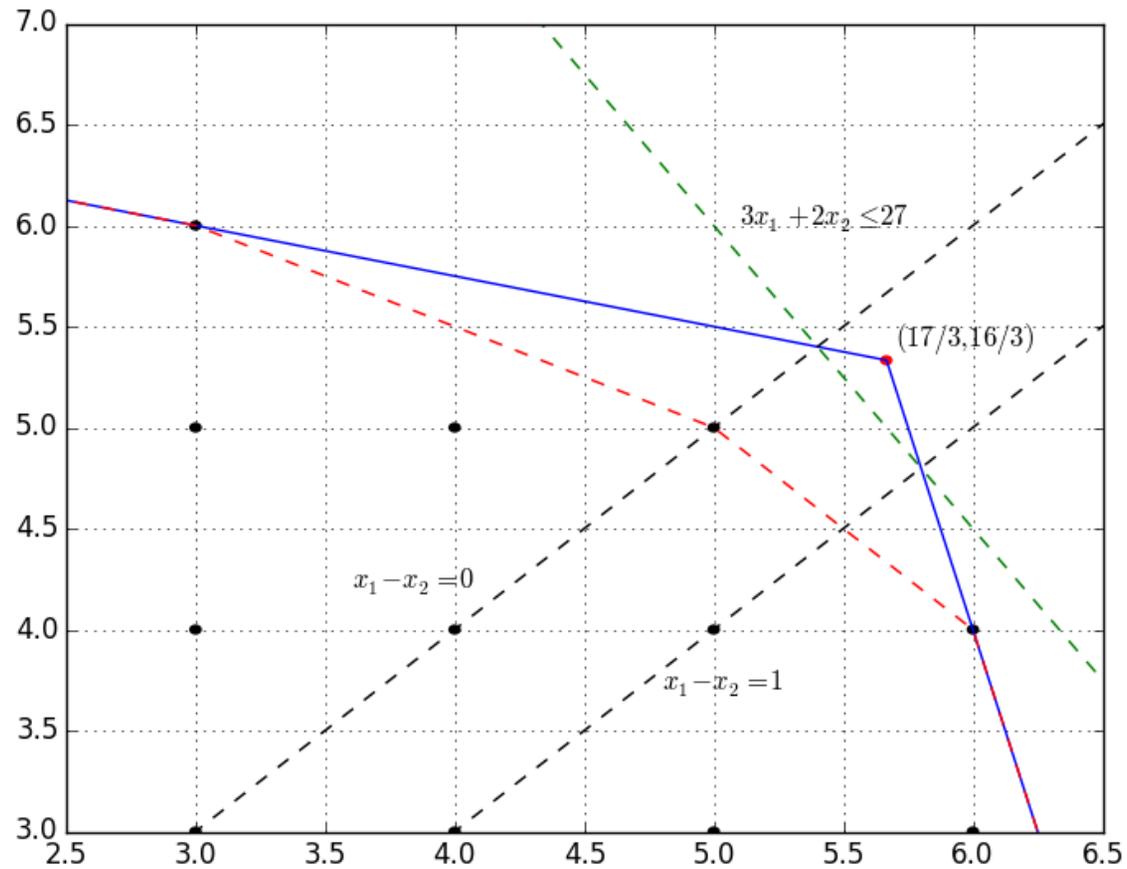


Figure 5: GMI Cut from row 2 as an intersection cut

GMI Cuts in Practice

Here is an example of the slow convergence sometimes seen in practice.

$$\begin{array}{ll} \min & 20x_1 + 15x_2 \\ & -2x_1 - 3x_2 \leq -5 \\ & -4x_1 - 2x_2 \leq -15 \\ & -3x_1 - 4x_2 \leq 20 \\ & 0 \leq x_1 \leq 9 \\ & 0 \leq x_2 \leq 6 \\ & x_1, x_2 \in \mathbb{Z} \end{array}$$

We will solve this using the naive implementation in CuPPy.

The Polyhedra in Example

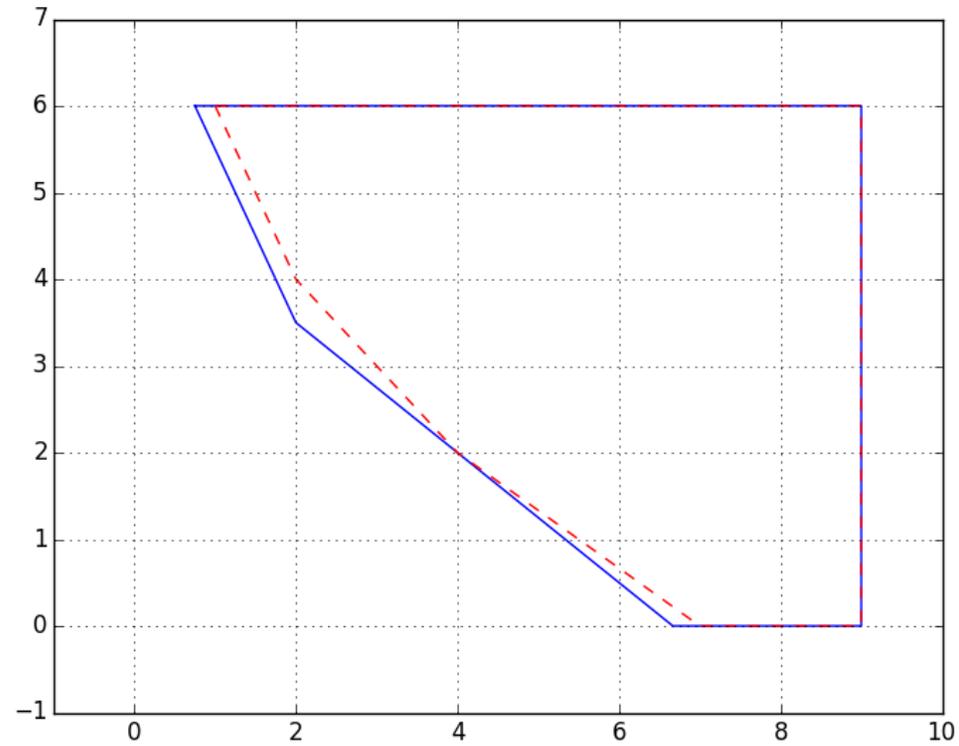


Figure 6: Feasible region of Example MILP

First Iteration

- The solution to the LP relaxation is $(2, 3.5)$.
- The tableau row in which x_2 is basic is

$$x_2 + 0.3s_2 - 0.4s_3$$

- Note that for purposes of illustration, we are explicitly included the bound constraints in the tableau.
- The GMI is

$$0.6s_2 + 0.8s_3 \geq 1$$

- In terms of the original variables, this is

$$-4.8x_1 - 4.4x_2 \leq -26$$

Second Iteration

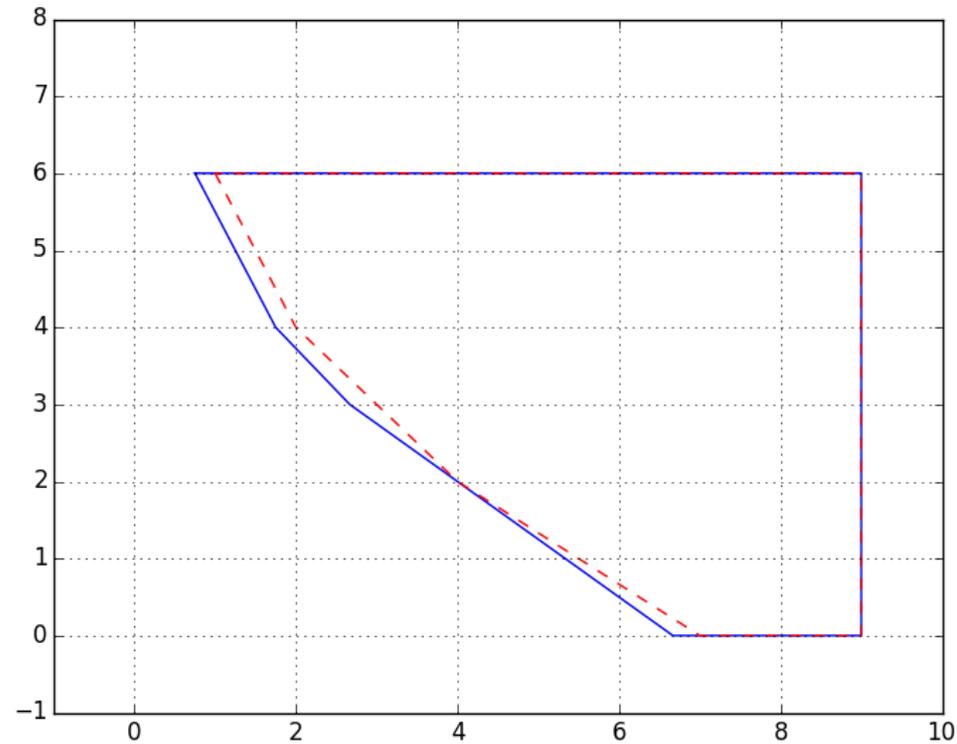


Figure 7: Feasible region of Example MILP after adding cut

The solution in the second iteration is $(1.75, 4)$ and the cut is $-10.4x_1 - 5.8667x_2 \leq -42.6667$.

Third Iteration

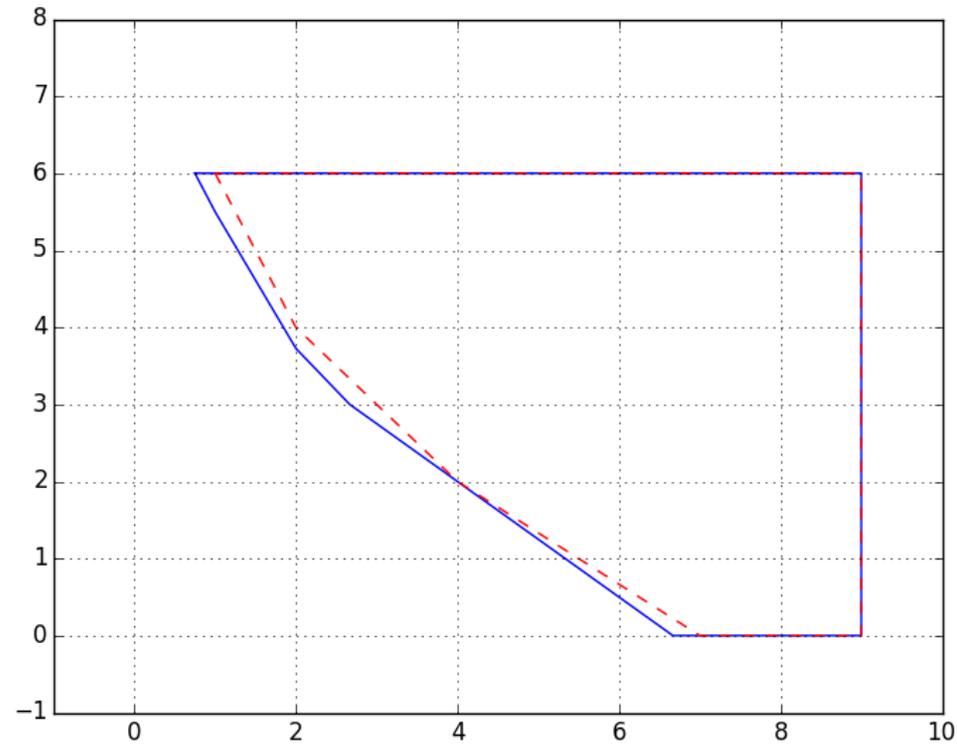


Figure 8: Feasible region of Example MILP after two cuts

The solution in the third iteration is $(2, 3.7273)$ and the cut is $-14.3x_1 - 11.7333x_2 \leq -73.3333$.

Further Iterations

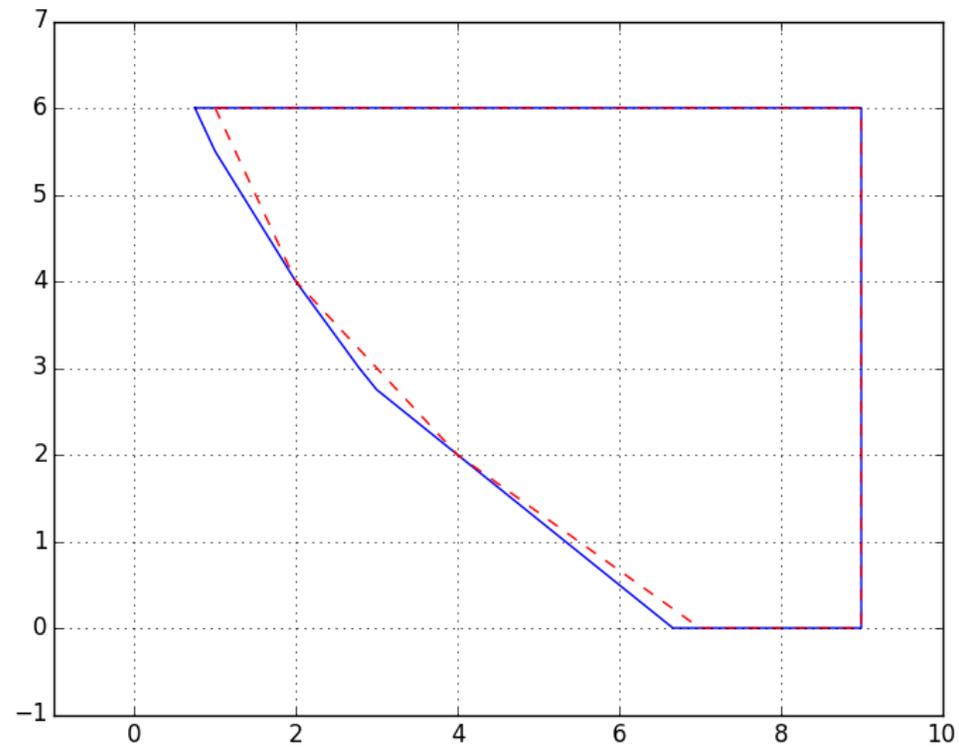


Figure 9: Feasible region of Example MILP after 100 cuts

Further Iterations

- Note the slow convergence rate.
- Not much progress is being made with each cut.
- After 100 iteration, the solution is $(1.9979, 4)$, which may be “close enough,” but would not be considered optimal by most solvers.
- It is surprising that such a small MILP would have such a high rank.
- This is at least partly due to numerical errors and the fact that our implementation is naive.
- We will delve further into these topics later in the course.