1. Consider applying Lagrangian relaxation to this familiar integer program.

\[
\begin{align*}
\text{max} & \quad 2x_1 + 5x_2 \\
\text{s.t.} & \quad 4x_1 + x_2 \leq 28 \\
& \quad x_1 + 4x_2 \leq 27 \\
& \quad x_1 - x_2 \leq 1 \\
& \quad x \in \mathbb{Z}^2_+ 
\end{align*}
\]  

(a) Explain graphically why if any two constraints are dualized, the optimal value of the Lagrangian dual is the same as that of the LP relaxation. You probably want to use GrUMPy for this.

(b) Find a different objective function for which the results from part (a) does not hold.

(c) Show graphically that if any single constraint is dualized, the optimal value of the Lagrangian dual is an improvement over that of the LP relaxation.

2. Consider the following budget-constrained assignment problem.

\[
\begin{align*}
\sum_{i \in N} x_{ij} &= 1 \text{ for } j \in M, \\
\sum_{j \in M} x_{ij} &= 1 \text{ for } i \in N, \\
\sum_{i \in N} \sum_{j \in N} a_{ij} x_{ij} &\leq b \\
0 &\leq x_{ij} \leq 1 \text{ for } \{i, j\} \in E, \text{ and} \\
x &\in \mathbb{Z}^E
\end{align*}
\]

(a) Discuss possible decompositions and compare the relative strengths of the resulting bounds, as well as the difficulty of the resulting subproblems.

(b) Generate some random instances and verify your results from the previous part empirically using DipPy.

3. Consider the problem

\[
\max \{ c^T x \mid A^i x \leq b^i, i = 1, 2, x \in \mathbb{Z}^n_+ \}.
\]
Suppose we reformulate as
\[
\max\{\alpha c^\top x^1 + (1 - \alpha) c x^2 \mid A_i x^i \leq b_i, i = 1, 2, x^1 = x^2, x \in \mathbb{Z}^n_+\}
\] (12)
for \(0 < \alpha < 1\). Show that the decomposition bound obtained by relaxing the constraint \(x^1 = x^2\) is
\[
\max\{c^\top x \mid x \in \text{conv}(S^1) \cap \text{conv}(S^2)\},
\] (13)
where
\[
S^i = \{x \in \mathbb{Z}^n_+ \mid A^i x \leq b^i\}. \tag{14}
\]

4. (a) Generate a few random MILPs and solve each instance \(m\) times, each time relaxing one additional constraint (you may use the MILP example from DipPy). Make a graph of how the bound, the number of nodes generated, and the solution time change as a function of the number of constraints relaxed. Explain your results by comparing them to your expectations and to theoretical predictions.

(b) Generate some random block-structured MILPs (again, you may use the MILP example from DipPy). Compare number of nodes generated and solution time for with and without decomposition. Analyze your results.