Homework 1
IE418 – Integer Programming
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Due September 9, 2014

1. (N&W I.1.1) Show that the integer program with irrational data

$$\max \{x_1 - \sqrt{2}x_2 : 1 \leq x_1 \leq \sqrt{2}x_2, x \in \mathbb{Z}^2\}$$

has no optimal solution, even though there exist feasible solutions with value arbitrarily close to zero.

2. (N&W I.1.3) An airline has fixed its daily timetable for flights between five cities. It now has the problem of scheduling the crews. There are certain legal limits in how much time each crew can work within 24-hour period. The problem is to propose a crew schedule using the minimum number of crews in which each flight leg is covered. Formulate a generic problem of this type as a set covering problem.

3. (N&W I.1.5) Integer and mixed-integer programming models are used on Wall Street to select bond portfolios. The idea is to pick a mix of bonds to maximize average yield subject to constraints on quality, length of maturity, industrial and government percentages, and total budget. Integrality arises because certain bonds only come in 100-unit lots. Formulate a model for this generic problem.

4. Create a model in PuLP for the two formulations of the uncapacitated lot-sizing problem given in lectures. Use each of your models to solve an instance of the ULS over six periods with demands \((6, 7, 4, 6, 3, 8)\), unit production costs \((3, 4, 3, 4, 4, 5)\), unit storage costs \((1, 1, 1, 1, 1, 1)\), set-up costs \((12, 15, 30, 23, 19, 45)\), and a maximum production capacity of 10 items per period. What are your observations?

5. Show that

$$X = \{ x \in \mathbb{B}^4 : 97x_1 + 32x_2 + 25x_3 + 20x_4 \leq 139 \}$$

$$= \{ x \in \mathbb{B}^4 : 2x_1 + x_2 + x_3 + x_4 \leq 3 \}$$

$$= \{ x \in \mathbb{B}^4 : x_1 + x_2 + x_3 \leq 2, \quad x_1 + x_2 + x_4 \leq 2, \quad x_1 + x_3 + x_4 \leq 2 \}$$

6. The Traveling Salesman Problem. We are given a set of nodes \(V = \{1, \ldots, n\}\) and a set of arcs \(A\). The nodes represent cities, and the arcs represent ordered pairs of cities between which direct travel is possible. For \((i, j) \in A, c_{ij}\) is the direct travel time from city \(i\) to city \(j\). The problem is to find a tour starting at city 1, that (a) visit each city exactly once and then returns to city 1 and (b) takes the least total travel time.
To formulate this problem, we introduce variables $x_{ij} = 1$ if $j$ immediately follows $i$ on the tour, $x_{ij} = 0$ otherwise. Hence:

$$x \in \mathbb{B}^{|A|}$$  \hspace{1cm} (3.11)

The requirement that each city is entered and left exactly once are stated as

$$\sum_{\{i : (i,j) \in A\}} x_{ij} = 1 \text{ for } j \in V$$  \hspace{1cm} (3.12)

and

$$\sum_{\{j : (i,j) \in A\}} x_{ij} = 1 \text{ for } i \in V$$  \hspace{1cm} (3.13)

One way to eliminate subtours is adding the set constraints 3.14. For any $U \subset V$ with $2 \leq |U| \leq |V| - 2$,

$$\sum_{(i,j) \in A : i \in U, j \in V \setminus U} x_{ij} \geq 1$$  \hspace{1cm} (3.14)

Hence the traveling salesman problem can be formulated as

$$\min \{ \sum_{(i,j) \in A} c_{ij} x_{ij} : x \text{ satisfies (3.11) – (3.14)} \}$$  \hspace{1cm} (3.15)

An alternative to the set of constraints 3.14 is

$$\sum (i,j) \in A : i \in U, j \in U x_{ij} \leq |U| - 1 \text{ for } 2 \leq |U| \leq |V| - 2,$$  \hspace{1cm} (3.16)

which also excludes all subtours but not tours.

(a) Prove formally that (3.15) in N&W is a valid formulation for the Traveling Salesman Problem.

(b) Prove that each constraint of the (3.16) is equivalent to constraints of the form (3.14).