Reading for This Lecture

• Chapter 4, Section 3-4
Optimality Conditions

Equality Constrained Problems
FJ with Equality Constraints

**Theorem 1.** Consider $S = \{x \in X : g_i(x) \leq 0, i \in [1, m], h_i(x) = 0, i \in [1, l]\}$ where $X$ is a nonempty open set in $\mathbb{R}^n$ and $g_i : \mathbb{R}^n \to \mathbb{R}, i \in [1, m]$, $h_i : \mathbb{R}^n \to \mathbb{R}, i \in [1, l]$. Given a feasible $x^* \in S$, set $I = \{i : g_i(x^*) = 0\}$. Assume that $f$ and $g_i$ are differentiable at $x^*$ for $i \in I$, $g_i$ is continuous at $x^*$ for $i \notin I$, $h_i$ is continuously differentiable. If $x^*$ is a local minimum, then there exists $\mu \in \mathbb{R}^m, v \in \mathbb{R}^l$ such that

$$
\mu_0 \nabla f(x^*) + \sum \mu_i \nabla g_i(x^*) + \sum v_i \nabla h_i(x^*) = 0
$$

$$
\mu_i g_i(x^*) = 0 \forall i \in [1, m]
$$

$$
\mu \geq 0
$$

$$
\mu \neq 0
$$
**Constraint Qualification**

- The same development applies here as with just inequality constraints.
- **Constraint qualification:** $\nabla g_i(x^*), i \in I$, and $\nabla h_i(x^*), i \in [1, l]$, are linearly independent.
- This CQ again implies $\mu_0 > 0$.
- We can hence derive similar KKT conditions for problems with equality constraints.
Convex Programs

- The KKT conditions are sufficient for *convex programs*:
  - \( f \) is convex
  - \( g_1, \ldots, g_m \) is convex
  - \( h_1, \ldots, h_l \) is linear

- The KKT conditions are necessary and sufficient for convex programs with all linear constraints.
Other Constraint Qualifications

• There are other (less restrictive) conditions that imply the necessity of the KKT conditions (Chapter 5).

• For convex programs, the *Slater condition* implies the necessity of the KKT conditions.
  – $\nabla h_i(x^*)$ are linearly independent.
  – there exists $x' \in S$ such that $g_i(x') < 0$, $\forall i \in I$. 
The Restricted Lagrangian

- Consider a given vector $x^* \in \mathbb{R}^n$.
- Define the *restricted Lagrangian* function with respect to $x^*$, $u^*$, and $v^*$ as

$$L(x) \equiv \Phi(x, u^*, v^*) \equiv f(x) + \sum_{i \in I} u^*_i g_i(x) + \sum v^*_i h_i(x)$$

where $I = \{i : g_i(x^*) = 0\}$.

- Note that dual feasibility is now equivalent to $\nabla L(x^*) = 0$. 


KKT Sufficient Conditions (2\textsuperscript{nd} Order)

• Theorem 2. Consider the problem to minimize $f : \mathbb{R}^n \to \mathbb{R}$ over $S = \{x \in X : g_i(x) \leq 0, i \in [1, m], h_i(x) = 0, i \in [1, l]\}$ where $X$ is a nonempty open set in $\mathbb{R}^n$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in [1, m]$, $h_i : \mathbb{R}^n \to \mathbb{R}$, $i \in [1, l]$. Assume $f$ and all constraint functions are twice differentiable.

Suppose $x^*$ is a KKT point with restricted Lagrangian function $L$.

- If $\nabla^2 L(x)$ is positive semi-definite $\forall x \in S$, then $x^*$ is a global minimum.
- If $\nabla^2 L(x)$ is positive semi-definite in a neighborhood of $x^*$, then $x^*$ is a local minimum.
- If $\nabla^2 L(x^*)$ is positive definite, then $x^*$ is a strict local minimum.
Strongly and Weakly Binding Constraints

- Consider a constrained optimization problem over a nonempty open set $X$ in $\mathbb{R}^n$ where the objective function and all the constraints are twice differentiable.

- Let $x^*$ be a KKT point with Lagrange multipliers $u^*$ and $v^*$.

- Inequality constraint $i$ is called weakly binding if $u^*_i = 0$ and strongly binding otherwise.

- Note that we could delete the weakly binding constraints and we would still have a KKT point.

- However, this might change the optimality status of the solution.