Advanced Mathematical Programming
IE417

Lecture 6

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Reading for This Lecture

- Chapter 4, Sections 1-2
Optimality Conditions

Unconstrained Problems
First-order Necessary Conditions

**Theorem 1.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be differentiable at \( x^* \). If there is a vector \( d \) such that \( \nabla f(x^*)^T d < 0 \), then there exists a \( \delta > 0 \) such that \( f(x^* + \lambda d) < f(x^*) \) for each \( \lambda \in (0, \delta) \).

**Corollary 1.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be differentiable at \( x^* \). If \( x^* \) is a local minimum, then \( \nabla f(x^*) = 0 \).

- The direction \( d \) is called a *descent direction*. 
Second-order Necessary Conditions

**Theorem 2.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable at $x^*$. If $x^*$ is a local minimum, then $\nabla f(x^*) = 0$ and $H(x^*)$ is positive semi-definite.
Sufficient Conditions

Theorem 3. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be twice differentiable at $x^*$. If $\nabla f(x^*) = 0$ and $H(x^*)$ is positive definite, then $x^*$ is a local minimum.

Theorem 4. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be pseudoconvex at $x^*$. Then $x^*$ is a global minimum, if and only if $\nabla f(x^*) = 0$.

Theorem 5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be infinitely differentiable. Then $x^*$ is a local minimum if and only if either $f^{(j)}(x^*) = 0 \ \forall j$, or else there exists an even $n$ such that $f^{(n)}(x^*) > 0$ while $f^{(j)}(x^*) = 0 \ \forall j < n$. 
Optimality Conditions

Inequality Constrained Problems
Feasible and Improving Directions

Definition 1. Let \( S \) be a nonempty set in \( \mathbb{R}^n \) and let \( x^* \in cl(S) \). The cone of feasible directions of \( S \) at \( x^* \) is given by

\[
D = \{ d : d \neq 0 \text{ and } x^* + \lambda d \in S, \forall \lambda \in (0, \delta), \exists \delta > 0 \}
\]

Definition 2. Let \( S \) be a nonempty set in \( \mathbb{R}^n \) and let \( x^* \in cl(S) \). Given a function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), the cone of improving directions of \( f \) at \( x^* \) is given by

\[
F = \{ d : f(x^* + \lambda d) < f(x^*), \forall \lambda \in (0, \delta), \exists \delta > 0 \}
\]
Characterizing Set F

Define $F_0 = \{d : \nabla f(x^*)^T d < 0\}$ and $F'_0 = \{d : d \neq 0, \nabla f(x^*)^T d \leq 0\}$. Then $F_0 \subseteq F \subseteq F'_0$.

**Theorem 6.** Let $S$ be a nonempty set in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be given. Consider the constrained optimization problem

$$\min f(x)$$

s.t. $x \in S$

If $x^*$ is a local optimal solution and $f$ is differentiable at $x^*$, then $F_0 \cap D$ is empty. Conversely,...
Characterizing Set D

• Consider the feasible region \( S = \{ x \in X : g_i(x) \leq 0, i \in [1, m] \} \) where \( X \) is a nonempty open set in \( \mathbb{R}^n \) and \( g_i : \mathbb{R}^n \to \mathbb{R}, i \in [1, m] \).

• Given a feasible \( x^* \in \mathbb{R}^n \), set \( I = \{ i : g_i(x^*) = 0 \} \).

• Assume that \( g_i \) is differentiable at \( x^* \) for \( i \in I \) and \( g_i \) is continuous at \( x^* \) for \( i \notin I \) and define \( G_0 = \{ d : \nabla g_i(x^*)^T d < 0 \ \forall i \in I \} \) and \( G'_0 = \{ d : d \neq 0, \nabla g_i(x^*)^T d \leq 0 \ \forall i \in I \} \). Then \( G_0 \subseteq D \subseteq G'_0 \).
More Optimality Conditions

Theorem 7. Let $X$ be a nonempty open set in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ be given. Consider the constrained optimization problem

$$\min f(x)$$
$$\text{s.t. } g_i(x) \leq 0$$
$$x \in X$$

If $x^*$ is a local optimal solution, then $F_0 \cap G_0$ is empty. Conversely, ...
Fritz-John Necessary Conditions

Theorem 8. Consider the feasible region \( S = \{x \in X : g_i(x) \leq 0, i \in [1, m]\} \) where \( X \) is a nonempty open set in \( \mathbb{R}^n \) and \( g_i : \mathbb{R}^n \to \mathbb{R}, i \in [1, m] \).

Given a feasible \( x^* \in S \), set \( I = \{i : g_i(x^*) = 0\} \). Assume that \( f \) and \( g_i \) are differentiable at \( x^* \) for \( i \in I \) and \( g_i \) is continuous at \( x^* \) for \( i \notin I \). If \( x^* \) is a local minimum, then there exists \( \mu \in \mathbb{R}^m \) such that

\[
\mu_0 \nabla f(x^*) + \sum \mu_i \nabla g_i(x^*) = 0
\]
\[
\mu_i g_i(x^*) = 0 \quad \forall i \in [1, m]
\]
\[
\mu \geq 0
\]
\[
\mu \neq 0
\]