Reading for This Lecture

- Chapter 10
Linear Programming

• We are given \( A \in \mathbb{R}^{m \times n}, \ b \in \mathbb{R}^m \). Assume \( S = \{ x \in \mathbb{R}^n \ \text{s.t.} \ \ Ax = b, \ x \geq 0 \} \) is bounded.

• We want to solve the LP

\[
\begin{align*}
\text{min } & \quad c^\top x \\
\text{s.t. } & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

• We need only consider the extreme points.
Characterization of Extreme Points

• Arrange the columns of $A$ such that $A = [B, N]$, where $B$ is a non-singular $n \times n$ matrix.

• Then $x$ is an extreme point of $S$ if and only if $x = [x_B, 0]$ where $x_B = B^{-1}b$ for some arrangement such that $B^{-1}b \geq 0$

• This implies that the number of extreme points is finite (but still potentially very large).
The Simplex Algorithm

• Note that $x_B = B^{-1}b - B^{-1}Nx_N$

• Hence, $c^\top x = c_B^\top x_B + c - N^\top x_N = c_B^\top B^{-1}b + (c_N^\top - c_B^\top B^{-1}N)x_N$

• So if $c_N^\top - c_B^\top B^{-1}N \geq 0$, we have found the optimal solution (why?)

• Otherwise, suppose some component of $c_N^\top - c_B^\top B^{-1}N$ is negative.

• How can we interpret this in terms of our framework of improving feasible directions?
The simplest rule for selecting an entering variables is to choose the one with the largest reduced cost.

However, we can do something smarter.

Choose the “steepest edge”, i.e., the one that forms the smallest angle with $- \nabla f(x_k) = -c$.

Let $\eta_j$ be the direction derived earlier for the edge corresponding to a non-basic variable $j$ entering the basis.

Note that $c^\top \eta_j$ is the corresponding reduced cost.

Instead of reduced costs, we look at $c^\top \eta_j / \| \eta_j \|$.
Implementing Steepest Edge Simplex Algorithm

- Steepest edge simplex algorithm
  - Compute $yB = c_B^\top$
  - Choose the column of $a_j$ of $N$ that maximizes $c_j - ya_j / \| \eta_j \|$.
  - Compute $Bd = a_j$
  - Find the largest $t$ such that $x_B^* - td \geq 0$
  - Set the value of $x_j$ to $t$ and the values of the basic variables to $x_B^* - td$.
  - Update the basis and the steepest edge norms.
Implementing the Algorithm

• Because of the computational effort required to update the steepest edge norms, these methods were not widely used until the 1990s.
• Recall that last semester we discussed methods for updating the basis efficiently after each iteration.
• There are also methods for efficiently updating the steepest edge norms.
• There are variations for the dual simplex.
• There are also variations in which the norms are approximated or where only a subset are considered.
Some Other Connections

- Notice that the reduced costs are also the values of the dual variables.
- This means that they are also equal to the KKT multipliers.
- Hence, each iteration of the simplex method essentially solves the equations given by KKT for the current basis.
- If the Lagrange multipliers/reduced costs/dual variables are positive, then we have a KKT point (optimal).
Reduced Gradient Methods

• We now consider the following problem

\[ \begin{align*} 
\min & \quad f(x) \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0 
\end{align*} \]

where \( A \in \mathbb{R}^{m \times n}, \ b \in \mathbb{R}^m, \ f : \mathbb{R}^n \to \mathbb{R} \).

• We assume

  – \( f \) is continuously differentiable,
  – any \( m \) columns of \( A \) are linearly independent, and
  – all extreme points of the feasible region have \( m \) strictly positive components.

• We use a generalization of ideas from the simplex algorithm.

• The biggest difference is that we can now have nonbasic variables that are not at their bounds.

• The \( m \) basic variables are chosen to be the “most positive” ones.
• The search direction can change the value of more than one basic variable.

• This method is guaranteed to converge to a KKT point.
Convex Simplex Method

• Note that a major difference between the reduced gradient method and the simplex algorithm is that the search direction is not restricted.

• Restricting ourselves to changing only one nonbasic variable at a time yields a method that behaves much like the simplex algorithm.

• This method is know as the convex simplex method, because it was originally applied to solve problem with convex objective functions.

• This method is also guaranteed to converge to a KKT point.

• Recall that if $f$ is convex, then the KKT conditions are necessary and sufficient.