Advanced Mathematical Programming
IE417

Lecture 2

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Reading for This Lecture

• Primary Reading
  – Chapter 2, Sections 1-3

• Secondary Reading
  – Chapter 1
  – Appendix A
Preliminaries
Real Vector Spaces

• A real vector space is a set $V$, along with
  
  – an addition operation that is closed, commutative, and associative.
  – an element $0 \in V$ such that $a + 0 = a$, $\forall a \in V$.
  – an additive inverse operation such that $\forall a \in V, \exists a' \in V$ such that $a + a' = 0$.
  – a closed, scalar multiplication operation such that $\forall \lambda, \mu \in \mathbb{R}, a, b \in V$
    * $\lambda(a + b) = \lambda a + \lambda b$
    * $(\lambda + \mu)a = \lambda a + \mu a$
    * $\lambda(\mu a) = (\lambda \mu)a$
    * $1a = a$
Norms on Vector Spaces

- A **norm** on a vector space is a function \( \| \cdot \| : V \rightarrow \mathbb{R} \) satisfying
  - \( \| v \| \geq 0 \ \forall v \in V \)
  - \( \| v \| = 0 \) if and only if \( v = 0 \)
  - \( \| v + w \| \leq \| v \| + \| w \| , \ \forall v, w \in V \)
  - \( \| \lambda v \| = |\lambda| \cdot \| v \| \)

- Norms are used for measuring the “size” of an object or the “distance” between two objects in a vector space.

- These are the normal properties you would expect such a measure to have.
Examples of Vector Spaces

- $\mathbb{R}^n$
- $\mathbb{Z}^n$
- $\mathbb{R}^{n \times n}$
- $\{y \in \mathbb{R}^m : Ax = y, \exists x \in \mathbb{R}^n\}$
- Unless otherwise noted, we will be dealing with $\mathbb{R}^n$
Matrix and Vector Norms

- Unless otherwise indicated, we will use the $L_2$ norm for vectors and the corresponding norm for matrices.

- We will denote this by $\| \cdot \|$.

- The $L_2$ norm for matrices is defined as follows:

$$
\| A \| = \max\{\| Ax \| / \| x \|, x \neq 0\}
$$

- Note the following properties:
  - $| x^T y | \leq \| x \| \cdot \| y \|$
  - $\| Ax \| \leq \| A \| \cdot \| x \|$
  - $\| AB \| \leq \| A \| \cdot \| B \|$
Types of Optimization Problems

• **Unconstrained Optimization**

\[
\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad x \in X,
\end{align*}
\]

• **Constrained Optimization**

\[
\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad g_i(x) \leq 0, \quad i = 1, \ldots, l, \\
& \quad h_i(x) = 0, \quad i = 1, \ldots, m, \\
& \quad x \in X
\end{align*}
\]
Constrained Optimization Problems

• If $f$, $g_i$, and $h_i$ are linear functions, then we have a *linear program* (by convention, $X$ is $\mathbb{R}^n$ in this case).

• If some of $f$, $g_i$, and $h_i$ are nonlinear functions, then we have a *nonlinear program* (again, $X$ is $\mathbb{R}^n$ by convention).

• If $X$ is a discrete set, then we have a *discrete optimization problem* (DOP).

• If $X = \mathbb{Z}^n$, then we have an *integer program* (this terminology usually refers only to linear models).
Some Terms

- Feasible point
- Ball of radius $\varepsilon$, denoted $N_\varepsilon(x)$
- (Strict) local minimum
- (Strict) local maximum
- (Strict) global minimum/maximum
Where We’re Going

- Given an optimization problem, the end goal is to determine a **globally optimal solution**.
- For nonlinear problems, we will sometimes have to settle for **local optima**.
- First, we will look at theoretical conditions that help us determine whether a given point is a local/global optimum.
- Then, we will look at algorithms which help us get there.
- We start by studying **convexity**.
Convex Analysis
Convex Sets

A set $S$ is *convex*

\[ x_1, x_2 \in S, \lambda \in [0, 1] \Rightarrow \lambda x_1 + (1 - \lambda)x_2 \in S \]

- If $y = \sum \lambda_i x_i$, where $\lambda_i \geq 0$ and $\sum \lambda_i = 1$, then $y$ is a *convex combination* of the $x_i$’s.
- If the positivity restriction on $\lambda$ is removed, then $y$ is an *affine combination* of the $x_i$’s.
- If we further remove the restriction that $\sum \lambda_i = 1$, then we have a *linear combination*. 
Convex Hull

- The *convex hull* of a set \( S \), \( \text{conv}(S) \), is the set of all convex combinations of the members of \( S \).

- The convex hull of \( S \) is
  - The smallest convex set containing \( S \)
  - The intersection of all convex sets containing \( S \)

- If \( S_1 \) and \( S_2 \) are convex sets, then so are the following:
  - \( S_1 \cap S_2 \)
  - \( S_1 + S_2 \)
  - \( S_1 - S_2 \)
Some More Terms

• We can similarly define the affine hull of a set $S$, $\text{aff}(S)$.

• A set of points $x_1, \ldots, x_k$ in $\mathbb{R}^n$ are affinely independent if $x_i \notin \text{aff}\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k\}$ $\forall i \in 1 \ldots k$.

• Note that $x_1, \ldots, x_k$ are affinely independent if and only if $x_2 - x_1, \ldots, x_n - x_1$ are linearly independent.

• A polytope is the convex hull of a set $S$ containing a finite number of points.

• If the set of points in $S$ are affinely independent, then the polytope is called a simplex and the points in $S$ are its vertices.
Carathéodory’s Theorem

**Theorem 1.** If $S$ is an arbitrary set in $\mathbb{R}^n$ and $x \in \text{conv}(S)$, then $x$ is the convex combination of at most $n + 1$ points.

**Idea of Proof:**
Closure and Interior

- A point $x$ is in the **closure** of a set $S$, denoted $cl(S)$, if $S \cap N_\varepsilon(x) \neq \emptyset, \forall \varepsilon > 0$.

- A set is **closed** if $S = cl(S)$.

- A point $x$ is in the **interior** of a set $S$, denoted $int(S)$, if $\exists \varepsilon > 0$ such that $N_\varepsilon(x) \subset S$.

- A set is **open** if $S = int(S)$.

- A point $x$ is on the **boundary** of a set $S$ if . . .

- A set $S$ is **bounded** if . . .

- A set $S$ is **compact** if it is closed and bounded.
Weierstrass’s Theorem

**Theorem 2.** Let $S$ be a nonempty, compact set, and let $f : S \to \mathbb{R}$ be continuous on $S$. Then there exists a solution to the optimization problem

$$\begin{align*}
\min & \ f(x) \\
\text{s.t.} & \ x \in S,
\end{align*}$$

**Idea of Proof:**