Graphs and Network Flows
IE411

Lecture 23

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Multi-Commodity Flow Problems

- Up until now, we have dealt only with flow problems that have implicitly involved just one *commodity*.
- In many real world problems, multiple commodities must be moved through a shared network.
- Supplies and demands are specified on a per-commodity basis.
- Capacities are shared between all commodities.

Assumptions

- Homogeneous goods
- No congestion
- Divisibility of good

Formal Definition

- Let $G = (N, A)$ be a directed network with a cost c_{ij} and a capacity u_{ij} associated with every arc $(i, j) \in A$.
- Associated with each node $i \in N$ and each commodity $k \in K$ is a number $b(i, k)$, which is the supply or demand of commodity k at node i.
- We let x_{ij}^k denote the amount of flow of commodity k sent on arc (i, j).
- The objective of the *multicommodity flow problem* is to determine a least cost way to move all commodities through the network in order to satisfy demands at each node subject to shared capacity constraints.

LP Formulation

$$
\begin{aligned}
\text{Minimize} \quad & z(x) = \sum_{(i,j)\in A, k\in K} c_{ij}^k x_{ij}^k & & & (1)\\
\text{subject to} \quad & \sum_{j:(i,j)\in A} x_{ij}^k - \sum_{j:(j,i)\in A} x_{ji}^k &= b(i,k) & \quad \forall i \in N,\ k \in K & (2)\\
& \sum_{k\in K} x_{ij}^k &\le u_{ij} & \quad \forall (i,j) \in A & (3)\\
& \sum_{k\in K} x_{ij}^k &\ge l_{ij} & \quad \forall (i,j) \in A & (4)\\
& 0 \le x_{ij}^k &\le u_{ij}^k & \quad \forall (i,j) \in A, k \in K & (5)
\end{aligned}
$$

Reduced Costs

- Recall that the optimality conditions for the minimum cost network flow problem involved the concept of *node potentials*.
- These essentially captured the cost of routing flow.
- With multiple commodities, there is competition for capacity among the commodities.
- We need separate potentials for each commodity and shared arc prices to capture the contribution to the cost of shipping commodity i of the capacity constraint on each arc.
  - w_{ij} will be the price associated with each arc.
  - $\pi^k(i)$ will be the potential associated with node i and commodity k.
- Then we have reduced costs $c_{ij}^{\pi,k} = c_{ij}^k + w_{ij} - \pi^k(i) + \pi^k(j)$.

Optimality Conditions

Theorem 1. *A feasible solution* y_{ij}^k *is an optimal solution for the multicommodity flow problem with* $u_{ij}^k = \infty$ *if and only if for some set of node potentials* $\pi^k, k \in K$ *and some set of arc prices* $w_{ij}, (i,j) \in A$ *the reduced costs and the flow values satisfy, for every* $(i,j) \in A$ *satisfy*

(a) $w_{ij}(\sum_{k\in K} y_{ij}^k - u_{ij}) = 0$ *for all* $(i,j) \in A$.

(b) *If* $c_{ij}^{\pi,k} \geq 0$ *for all* $(i,j) \in A, k \in K$

(c) $c_{ij}^{\pi,k} y_{ij}^k = 0$ *for all* $(i,j) \in A, k \in K$

Decomposition by Commodity

- The role of the arc prices is to assign a cost to flow that is independent of commodity.
- Suppose we have optimal arc prices and solve a minimum cost flow problem with each commodity separately
  - No arc capacities
  - Cost of each arc adjusted for all commodities by w_{ij}.
- It turns out that the flows that are optimal for the complete multicommodity flow problem will also be optimal for the decomposed flow problems with the adjusted costs!
- This is easy to show.

Sequential Solution Approach

- This means that we can use a sequential approach to solving this problem.
- First, we try to find optimal arc prices.
- Then we try to find corresponding node potentials by solving minimum cost flow problems.
- If the flows obtained in that way do not satisfy joint capacity constraints, then we may go back and adjust the arc prices.

Lagrangian Relaxation Approach

- We can calculate optimal arc costs using Lagrangian relaxation and subgradient optimization.
- The constraints being relaxed are the joint capacity constraints.
- The subgradient update formula becomes

$$w_{ij}^{q+1} = \left[w_{ij}^{q} + \theta_q \left(\sum_{k \in K} y_{ij}^{k} - u_{ij} \right) \right]^{+}$$

Path Formulation

- Another way to approach solution of multi-commodity flow problems is to use a *path formulation*.
- The concept is to have a variable for each possible path that each commodity can take.
- We assume that each commodity k has a single source node s^k and a single sink node t^k.
- Let $\mathcal{P}^k$ be the collection of all paths from s^k to t^k for commodity k.
- With each path P in the union of all paths for all commodities, we have a decision variable $f(P)$ that denotes the flow on path P.
- We then get an (apparently) very simple reformulation of the problem in terms of these new variables.

Path Formulation

$$\text{Minimize} \quad z(x) = \sum_{k \in K} \sum_{P \in \mathcal{P}^k} c^k(P) f(P) \tag{6}$$

$$\text{subject to} \quad \sum_{k \in K} \sum_{P \in \mathcal{P}^k} \sum_{(i,j) \in P} \delta_{ij}(P) f(P) \le u_{ij} \quad \forall (i,j) \in A \tag{7}$$

$$\sum_{P \in \mathcal{P}^k} f(P) = d^k \quad \forall k \in K \tag{8}$$

$$f(P) \ge 0 \quad \forall k \in K, P \in \mathcal{P}^k \tag{9}$$

where $\delta_{ij}(P)$ takes value one if arch (i,j) is in path P and 0 otherwise.

Connection with Arc Formulation

- Notice that we did not need to enforce the flow balance constraints because they are captured in the definition of the path variables.
- We can map solutions from the path formulation back to the arc formulation as follows:

$$x_{ij}^k = \sum_{P \in \mathcal{P}^k} \delta_{ij}(P) f(P)$$

- We are implicitly using this mapping to enforce capacity constraints by arc.
- The problem with this formulation is that the number of variables is very large.
- We can again use decomposition to solve the problem by generating "interesting" paths on the fly.

Reduced Costs

- With this formulation, we have a slightly different set of "prices."
- The constraints linking the path variables specify whether the total flow is equal to the demand for each commodity.
- We need a price for each commodity, as well as a price for each arc.
- Therefore, we have
  - an arc price w_{ij} associated with each arc $(i, j) \in A$, and
  - a commodity price σ^k associated with each commodity.
- Then we can define reduced costs per path as

$$c_P^{\sigma,w} = c^k(P) + \sum_{(i,j) \in P} w_{ij} - \sigma_k$$

- With this definition of reduced costs, we can derive another set of optimality conditions.

Path Flow Optimality Conditions

Theorem 2. *The commodity path flows* $f(P)$ *are optimal for the path flow formulation of the multicommodity flow problem with* $u_{ij}^k = \infty$ *if and only if for some set of commodity prices* $\sigma^k, k \in K$ *and some set of arc prices* $w_{ij}, (i,j) \in A$ *the reduced costs and the flow values satisfy*

(a) $w_{ij}(\sum_{k \in K, P \in \mathcal{P}^k} \delta_{ij}(P) f(P) - u_{ij}) = 0$ *for all* $(i,j) \in A$.

(b) *If* $c_P^{\sigma,k} \geq 0$ *for all* $P \in \mathcal{P}^k, k \in K$

(b) *If* $c_P^{\sigma,k} f(P) = 0$ *for all* $P \in \mathcal{P}^k, k \in K$

Generating Paths

- For a fixed set of prices, the reduced cost of a path is simply

$$c_P^{\sigma,w} = \sum_{(i,j)\in P} (c_{ij}^k + w_{ij}) - \sigma_k$$

- This is the cost of the path with respect to "reduced arc prices" and adjusted by the commodity price.
- We can therefore easily determine the path with the smallest reduced cost for each commodity by solving a shortest path problem.
- This leads to a decomposition algorithm very similar to the previous one in which we adjust arc and commodity prices.