Graphs and Network Flows
IE411

Lecture 16

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Outline

• Minimum Cost Flow Problems
  – LP Formulation
  – Optimality Conditions
  – Cycle Canceling Algorithm
**LP Formulation**

Let $G = (N, A)$ be a directed network with a cost $c_{ij}$ and a capacity $u_{ij}$ associated with every arc $(i, j) \in A$.

Associated with each node $i \in N$ is a number $b(i)$; we refer to node $i$ as a *supply* node if $b(i) > 0$ and as a *demand* node if $b(i) < 0$.

Let $x_{ij}$ denote the amount of flow sent on arc $(i, j)$.

The objective of the *minimum cost flow problem* is to determine a least cost shipment of a commodity through a network in order to satisfy demands at certain nodes from available supplies at other nodes.
Minimize \[ z(x) = \sum_{(i,j) \in A} c_{ij} x_{ij} \] (1)

subject to \[ \sum_{j:(i,j) \in A} x_{ij} - \sum_{j:(j,i) \in A} x_{ji} = b(i) \quad \forall i \in N \] (2)

\[ x_{ij} \leq u_{ij} \quad \forall (i,j) \in A \] (3)

\[ x_{ij} \geq l_{ij} \quad \forall (i,j) \in A \] (4)
Relationship to Shortest Path and Maximum Flow

• What changes are required to model the shortest path problem?

• What changes are required to model the maximum flow problem?
Assumptions

A1. All data (cost, supply/demand, capacity) are integer.

A2. The network is directed.

A3. The supplies/demands at the nodes satisfy $\sum_{i \in N} b(i) = 0$, and there is a feasible solution.

A4. $G$ contains an uncapacitated directed path between every pair of nodes.

A5. All arc costs are non-negative.
**Residual Network** $G(x)$

Replace each arc $(i, j) \in A$ by two arcs:
- $(i, j)$ with cost $c_{ij}$ and residual capacity $r_{ij} = u_{ij} - x_{ij}$
- $(j, i)$ with cost $-c_{ij}$ and residual capacity $r_{ji} = x_{ij}$
Optimality Conditions

• Recall the Shortest Path Optimality Conditions...

• Why are they useful?
  – Provide simple validity check
  – Suggest an algorithm
  – Provide mechanism for establishing correctness of an algorithm
Min Cost Flow Optimality Conditions

We will consider three (equivalent) optimality conditions.

- Negative Cycle Optimality Conditions
- Reduced Cost Optimality Conditions
- Complementary Slackness Optimality Conditions
Negative Cycle Optimality Conditions

Theorem 1. [9.1] A feasible solution $x^*$ is an optimal solution of the minimum cost flow problem if and only if the residual network $G(x^*)$ contains no negative cost (directed) cycle.

Proof:
1. Show that if a feasible solution $x^*$ is an optimal solution of the minimum cost flow problem, then $G(x^*)$ contains no negative cost (directed) cycle.

2. Show that if $G(x^*)$ contains no negative cost (directed) cycle, then $x^*$ is optimal.
Cycle-Canceling Algorithm

- Maintains a feasible solution; attempts to improve objective function value
- Establishes a feasible flow to start
- Iteratively finds a negative cost directed cycle in residual networks and augments flow along cycle
- Terminates when residual network contains no negative cost directed cycle
Generic Cycle-Canceling Algorithm

**algorithm cycle-canceling** (Klein, 1967)

begin

establish a feasible flow $x$ in the network

while $G(x)$ contains a negative cycle **do**

identify a negative cycle $W$

$\delta = \min\{r_{ij} : (i, j) \in W\}$

augment $\delta$ units in cycle $W$ and update $G(x)$

end


Example
Complexity of Generic Cycle-Canceling Algorithm

- How many iterations?

- How much work during each iteration?
Implementations of Cycle-Canceling Algorithm

- Negative Cycle with Maximum Improvement
  - $O(m \log (mCU))$ – Barahona and Tardos (1989)
  - Note: Max Improvement Cycle is NP-Complete

- Negative Cycle with Minimum Mean Cost
  - Goldberg and Tarjan (1988)
  - mean cost of a cycle $W = (\sum_{(i,j) \in W} c_{ij})/|W|
  - Identify Minimum Mean Cycle in $O(nm)$ or $O(\sqrt{n} \ m \log(nC))$
  - Min Cost Flow $O(\min\{nm\log(nC), nm^2\log(n)\})$ iterations
Integrality Property

Theorem 2. [9.10] If all arc capacities and node supplies/demands are integer, then the minimum cost flow problem always has an integer minimum cost flow.

Proof: