1. This question involves implementing methods of solving systems of equations in Python.

(a) Implement a method in Python using numpy for finding the LUP decomposition of a matrix in place, as discussed in class. You should include methods for scaling and pivoting, as discussed in class, along with flags for disabling these methods. You should also include an option for limiting the precision of the calculations to a specified number of digits (this is to make it simpler to explore the effects on accuracy of the implemented methods).

(b) Implement a method for solving a system of equation using your method of LUP decomposition from the previous question.

(c) Extend your solver by adding a method for doing iterative improvement. You can use the option to restrict the precision of your computation during the solution of the systems in order to “simulate” the calculation of the residuals at a higher level of precision. Alternatively, you can also use the GMPy (a library for doing arithmetic in arbitrary precision) to do the calculation of the residual (this is more like the way it would probably be done in practice).

2. This question is to test your implementations from the previous question.

(a) Generate some random matrices, compute their condition numbers (you might want to discard matrices whose condition numbers are not what you want for your experiments), and try to verify that in the absence of scaling, changes to the solution vector resulted from changes to the right-hand side vector are affected by the condition number of the matrix roughly as expected. You might want to solve your systems with numpy’s built-in solver as a benchmark.

(b) Experiment with the effect of applying pivoting and scaling on the accuracy of solutions you obtain from your solver. Again, you could use numpy’s built-in solver as a benchmark.

(c) Verify the results on convergence rate and the increase in precision of solution expected in each iteration of iterative improvement, as discussed in lecture.

3. Let

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

be a non-singular matrix. Define

\[ \sigma = \frac{a^2 + b^2 + c^2 + d^2}{2|ad - bc|} \]

Prove that \( \text{cond}(A) = \sigma + \sqrt{\sigma^2 - 1} \).
4. Let $x_1$ be an approximate solution of the linear system $Ax = b$, where $b \neq 0$. Let $e_1 = x_1 - A^{-1}b$ and $r_1 = b - Ax_1$. The relative error in $x_1$ is then defined as

$$\rho_x = \frac{\|e_1\|}{\|A^{-1}b\|}$$

and the corresponding relative residual is

$$\rho_r = \frac{\|r_1\|}{\|b\|}.$$

Show that

$$\frac{1}{\text{cond}(A)} \leq \frac{\rho_x}{\rho_r} \leq \text{cond}(A)$$