Introduction to Mathematical Programming
IE406

Lecture 3

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Reading for This Lecture

• Bertsimas 2.1-2.2
From Last Time

• Recall the Two Crude Petroleum example.

• In the example, the optimal solution was a “corner point.”

• We saw that the following are possible outcomes of solving an optimization problem:

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• In fact, we will see that these are the only possibilities.

• We will also see that when there is an optimal solution and at least one “corner point,” there is an optimal solution that is a “corner point.”
Some Definitions

Definition 1. A polyhedron is a set of the form \( \{ x \in \mathbb{R}^n | Ax \geq b \} \), where \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \).

Definition 2. A set \( S \subseteq \mathbb{R}^n \) is bounded if there exists a constant \( K \) such that \( |x_i| < K \) \( \forall x \in S, \forall i \in [1, n] \).

Definition 3. Let \( a \in \mathbb{R}^n \) and \( b \in \mathbb{R} \) be given.

- The set \( \{ x \in \mathbb{R}^n | a^\top x = b \} \) is called a hyperplane.
- The set \( \{ x \in \mathbb{R}^n | a^\top x \geq b \} \) is called a half-space.

Notes:
Convex Sets

Definition 4. A set $S \subseteq \mathbb{R}^n$ is convex if $\forall x, y \in S$ and $\lambda \in \mathbb{R}$ with $0 \leq \lambda \leq 1$, we have $\lambda x + (1 - \lambda)y \in S$.

Definition 5. Let $x^1, \ldots, x^k \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^k$ be given such that $\lambda^\top 1 = 1$.

1. The vector $\sum_{i=1}^{k} \lambda_i x^i$ is said to be a convex combination of $x^1, \ldots, x^k$.
2. The convex hull of $x^1, \ldots, x^k$ is the set of all convex combinations of these vectors.

Notes:
Properties of Convex Sets

The following properties can be derived from the definitions:

- The intersection of convex sets is convex.
- Every polyhedron is a convex set.
- The convex combination of a finite number of elements of a convex set also belongs to the set.
- The convex hull of a finite number of vectors is a convex set.

How do we prove each of these?
Aside: Mathematical Proofs

- A mathematical proof shows the correctness of a given statement based on known definitions, axioms, and previously proven statements.
- Most proofs are for statements of the form $A \Rightarrow B$ where $A$ and $B$ are both statements.
- **Example**: “If $x > 2$ is a real number, then there exists a real number $y < 0$ such that $x = \frac{2y}{1+y}$.”
- **Proof**:

- What are $A$ and $B$ in this example?
Mathematical Proofs: Quantifying Variables

- **Quantifying** is specifying from which set and for which values of a variable a statement is true.

- **Example**: “For all real numbers $x$ and $y$, $(x + y)^2 = x^2 + 2xy + y^2$.”

- This specifies that $x$ and $y$ can have any real value.

- **Example**: “For all real numbers $x \geq 0$, $x = |x|$.”

- This specifies that the statement is true for nonnegative values of $x$. 

Mathematical Proofs: Types of Quantifiers

• Universal Quantifiers
  – Statements that include “for all” or “for every.”
  – **Example**: “For all real numbers $x$, $\cos^2 x + \sin^2 x = 1$.”

• Existential Quantifiers
  – Statements that include “there exists” or “there is.”
  – **Example**: “For every real number $0 \leq x \leq 1$, there exists a real number $0 \leq y \leq \frac{\pi}{2}$ such that $\sin(y) = x$.”

• **Notation**: $\forall$ means “for all” and $\exists$ means “there exists”.

• **Example**: “$\forall x \in \mathbb{R}$ such that $0 \leq x \leq 1$, $\exists y \in \mathbb{R}$ such that $0 \leq y \leq \frac{\pi}{2}$ and $\sin(y) = x$. ”
Mathematical Proofs: Proofs with Universal Quantifiers

• To prove something about a universally quantified statement, first let an arbitrary set element be given.

• Example: “If $C \in \mathbb{R}^{n \times n}$ and $\det(C) \neq 0$, then $\exists C^{-1} \in \mathbb{R}^{n \times n}$ such that $CC^{-1} = I$.”

• Start of Proof: “Let an arbitrary matrix $C \in \mathbb{R}^{n \times n}$ be given such that $\det(C) \neq 0$...”

• Now prove that statement is true for the given element.

• Since the element was arbitrary, this proves the original statement.
If you are trying to prove something about an existentially quantified variable, the proof is often *constructive*.

The proof gives a technique for constructing an element of the set with the given property.

**Example:** “If $C \in \mathbb{R}^{n \times n}$ and $\det(C) \neq 0$, then $\exists C^{-1} \in \mathbb{R}^{n \times n}$ such that $CC^{-1} = I$.”

**Proof Technique:** Construct $C^{-1}$. 
Mathematical Proofs: Choosing an Element

- If you know from a previous theorem that an element of a set with a particular property exists, then you may “choose” it.

- Example: “Let $r$, a positive rational number be given. Then we may choose natural numbers $p$ and $q$ such that $r = \frac{p}{q}$.”

- This can be especially useful in constructive proofs.
Mathematical Proofs: Proof Techniques

- Prove the contrapositive.
- Proof by contradiction.
- Proof by induction.
- Proof by cases.
- Other types of proofs
  - Uniqueness proofs.
  - Either/or proofs.
  - If and only if proofs.
Let’s prove the following:

**Proposition 1.** The intersection of convex sets is convex.

**Proof:**

**Proposition 2.** Every polyhedron is convex.

**Proof:**
Extreme Points and Vertices

Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a given polyhedron.

**Definition 6.** A vector $x \in \mathcal{P}$ is an **extreme point** of $\mathcal{P}$ if $\forall y, z \in \mathcal{P}, \lambda \in (0, 1)$ such that $x = \lambda y + (1 - \lambda)z$.

**Definition 7.** A vector $x \in \mathcal{P}$ is a **vertex** of $\mathcal{P}$ if $\exists c \in \mathbb{R}^n$ such that $c^\top x < c^\top y \forall y \in \mathcal{P}, x \neq y$.

**Notes:**
A Little Linear Algebra Review

Definition 8. A finite collection of vectors \( x_1, \ldots, x_k \in \mathbb{R}^n \) is linearly independent if the unique solution to \( \sum_{i=1}^{k} \lambda_i x_i = 0 \) is \( \lambda_i = 0, i = 1, \ldots, k \). Otherwise, the vectors are linearly dependent.

Let \( A \) be a square matrix. Then, the following statements are equivalent:

- The matrix \( A \) is invertible.
- The matrix \( A^\top \) is invertible.
- The determinant of \( A \) is nonzero.
- The rows of \( A \) are linearly independent.
- The columns of \( A \) are linearly independent.
- For every vector \( b \), the system \( Ax = b \) has a unique solution.
- There exists some vector \( b \) for which the system \( Ax = b \) has a unique solution.
A Little More Linear Algebra Review

Definition 9. A nonempty subset \( S \subseteq \mathbb{R}^n \) is called a subspace if \( \alpha x + \gamma y \in S \) \( \forall x, y \in S \) and \( \forall \alpha, \gamma \in \mathbb{R} \).

Definition 10. A linear combination of a collection of vectors \( x^1, \ldots, x^k \in \mathbb{R}^n \) is any vector \( y \in \mathbb{R}^n \) such that \( y = \sum_{i=1}^{k} \lambda_i x^i \) for some \( \lambda \in \mathbb{R}^k \).

Definition 11. The span of a collection of vectors \( x^1, \ldots, x^k \in \mathbb{R}^n \) is the set of all linear combinations of those vectors.

Definition 12. Given a subspace \( S \subseteq \mathbb{R}^n \), a collection of linearly independent vectors whose span is \( S \) is called a basis of \( S \). The number of vectors in the basis is the dimension of the subspace.
Subspaces and Bases

- A given subspace has an infinite number of bases.
- Each basis has the same number of vectors in it.
- If \( S \) and \( T \) are subspaces such that \( S \subset T \subset \mathbb{R}^n \), then a basis of \( S \) can be extended to a basis of \( T \).
- The span of the columns of a matrix \( A \) is a subspace called the column space or the range, denoted \( \text{range}(A) \).
- The span of the rows of a matrix \( A \) is a subspace called the row space.
- The dimensions of the column space and row space are always equal. We call this number \( \text{rank}(A) \).
- Clearly, \( \text{rank}(A) \leq \min\{m, n\} \). If \( \text{rank}(A) = \min\{m, n\} \), then \( A \) is said have full rank.
- The set \( \{x \in \mathbb{R}^n | Ax = 0\} \) is called the null space of \( A \) (denoted null\( (A) \)) and has dimension \( n - \text{rank}(A) \).
Some Conventions

If not otherwise stated, the following conventions will be followed for lecture slides during the course:

- $\mathcal{P}$ will denote a polyhedron contained in $\mathbb{R}^n$.
- $A$ will denote a matrix of dimension $m$ by $n$.
- $b$ will denote a vector of dimension $m$.
- $x$ will denote a vector of dimension $n$.
- $c$ will denote a vector of dimension $n$.
- $\mathcal{P}$ will either be defined in standard form ($\{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$) or inequality form ($\{x \in \mathbb{R}^n | Ax \geq b\}$).
- We will usually be minimizing.