Reading for This Lecture

• Primary Reading
  – Bertsimas 1.1-1.2, 1.4-1.5

• Supplementary Reading
  – Bertsimas 1.3
  – Operations Research Methods and Models
  – Model Building in Mathematical Programming
Review from Last Time

- Recall that a mathematical model consists of:
  - Decision variables (with domains)
  - Constraints (functions of the variables with domains)
  - Objective Function (maximize or minimize)
  - Parameters and Data

The general form of a *mathematical programming model* is:

\[
\begin{align*}
\min \text{ or } \max & \quad f(x_1, \ldots, x_n) \\
\text{s.t.} & \quad g_i(x_1, \ldots, x_n) \begin{cases} \leq \end{cases} b_i \\
& \quad g_i(x_1, \ldots, x_n) \begin{cases} = \end{cases} b_i \\
& \quad g_i(x_1, \ldots, x_n) \begin{cases} \geq \end{cases} b_i \\
(x_1, \ldots, x_n) & \in X
\end{align*}
\]

where $X$ may be a discrete set.
Example of a Mathematical Program: The Diet Problem

- **Goal**: Choose the cheapest menu satisfying nutritional requirements.

- What is the **input data**?

- What is the **formulation** in words?
Critique of the Model

What are the possible problems with this model?
A Little History

- **George Dantzig** is considered to be the father of linear programming.
- The diet problem was one of the first applications of linear programming.
- It took *120 man-days* to solve a problem with 9 constraints and 77 variables by hand!
- Later, Dantzig tried to lose weight by designing his own diet.
  - The first solution he came up with contained several gallons of vinegar.
  - After deleting vinegar from the list of foods, the new solution contained approximately *200 bouillon cubes*.
  - This illustrates one of the potential hazards of math modeling.
Representing Math Models: Vectors and Matrices

• An \( m \times n \) matrix is an array of \( mn \) real numbers:

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]

• \( A \) is said to have \( n \) columns and \( m \) rows.

• An \( n\)-dimensional column vector is a matrix with one column.

• An \( n\)-dimensional row vector is a matrix with one row.

• By default, a vector will be considered a column vector.

• The set of all \( n\)-dimensional vectors will be denoted \( \mathbb{R}^n \).

• The set of all \( m \times n \) matrices will be denoted \( \mathbb{R}^{m \times n} \).
Matrices

• The transpose of a matrix $A$ is

$$A^\top = \begin{bmatrix}
  a_{11} & a_{21} & \cdots & a_{m1} \\
  a_{12} & a_{22} & \cdots & a_{m2} \\
  \vdots & \vdots & & \vdots \\
  a_{1n} & a_{2n} & \cdots & a_{mn}
\end{bmatrix}$$

• If $x, y \in \mathbb{R}^n$, then $x^\top y = \sum_{i=1}^{n} x_i y_i$.

• This is called the inner product of $x$ and $y$.

• If $A \in \mathbb{R}^{m \times n}$, then $A_j$ is the $j^{th}$ column, and $a_j^\top$ is the $j^{th}$ row.

• If $A \in \mathbb{R}^{m \times k}, B \in \mathbb{R}^{k \times n}$, then $[AB]_{ij} = a_i^\top B_j$. 
Linear Functions

- A **linear function** \( f : \mathbb{R}^n \to \mathbb{R} \) is a weighted sum, written as

\[
f(x_1, \ldots, x_n) = \sum_{i=1}^{n} c_i x_i
\]

for given coefficients \( c_1, \ldots, c_n \).

- We can write \( x_1, \ldots, x_n \) and \( c_1, \ldots, c_n \) as vectors \( x, c \in \mathbb{R}^n \) to obtain:

\[
f(x) = c^\top x
\]

- In this way, a linear function can be represented simply as a vector.

- We will consider only models defined by linear functions.
Back to the Diet Problem

- How do we write the diet problem mathematically?
- What are the decision variables?
- What is the objective function?
- What are the constraints?
Putting It All Together

• Using matrix notation, we can write our current formulation as

\[
\begin{align*}
\min & \quad c^\top x \\
\text{s.t.} & \quad l \leq Ax \leq u
\end{align*}
\]

• What are we missing?
Linear Programs

- What we have just seen is an example of a linear program.

- In general, we can write a linear program as

  \[
  \begin{align*}
  \text{minimize} & \quad c^\top x \\
  \text{s.t.} & \quad a_i^\top x \geq b_i \quad \forall i \in M_1 \\
  & \quad a_i^\top x \leq b_i \quad \forall i \in M_2 \\
  & \quad a_i^\top x = b_i \quad \forall i \in M_3 \\
  & \quad x_j \geq 0 \quad \forall j \in N_1 \\
  & \quad x_j \leq 0 \quad \forall j \in N_2
  \end{align*}
  \]

- This in turn can be written equivalently as

  \[
  \begin{align*}
  \text{minimize} & \quad c^\top x \\
  \text{s.t.} & \quad Ax \geq b
  \end{align*}
  \]

- How do we do this?
Standard Form

• To solve a linear program, it is convenient to put it in the following standard form:

\[ \min c^\top x \]

\[ s.t. \quad Ax = b \]

\[ x \geq 0 \]

• How do we do this?
Two Crude Petroleum Example

- Two Crude Petroleum distills crude from two sources:
  - Saudi Arabia
  - Venezuela

- They have three main products:
  - Gasoline
  - Jet fuel
  - Lubricants

- Yields

<table>
<thead>
<tr>
<th></th>
<th>Gasoline</th>
<th>Jet fuel</th>
<th>Lubricants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Saudi Arabia</td>
<td>0.3 barrels</td>
<td>0.4 barrels</td>
<td>0.2 barrels</td>
</tr>
<tr>
<td>Venezuela</td>
<td>0.4 barrels</td>
<td>0.2 barrels</td>
<td>0.3 barrels</td>
</tr>
</tbody>
</table>
Two Crude Petroleum Example (cont.)

• Availability and cost

<table>
<thead>
<tr>
<th></th>
<th>Availability</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>Saudi Arabia</td>
<td>9000 barrels</td>
<td>$20/barrel</td>
</tr>
<tr>
<td>Venezuela</td>
<td>6000 barrels</td>
<td>$15/barrel</td>
</tr>
</tbody>
</table>

• Production Requirements (per day)

<table>
<thead>
<tr>
<th></th>
<th>Gasoline</th>
<th>Jet fuel</th>
<th>Lubricants</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2000 barrels</td>
<td>1500 barrels</td>
<td>500 barrels</td>
</tr>
</tbody>
</table>

• **Objective**: Minimize production cost.
Modeling the Two Crude Production Problem

- What are the decision variables?
- What is the objective function?
- What are the constraints?
Linear Programming Formulation of Two Crude Example

- This yields the following LP formulation:

\[
\begin{align*}
\text{min} & \quad 20000x_1 + 15000x_2 \\
\text{s.t.} & \quad 0.3x_1 + 0.4x_2 \geq 2.0 \\
& \quad 0.4x_1 + 0.2x_2 \geq 1.5 \\
& \quad 0.2x_1 + 0.3x_2 \geq 0.5 \\
& \quad 0 \leq x_1 \leq 9 \\
& \quad 0 \leq x_2 \leq 6 
\end{align*}
\]

- How can we solve this problem?

- What are the possible outcomes of solving such a problem?