Reading for This Lecture

- Bertsimas 4.8-4.9
**Polyhedral Cones**

**Definition 1.** A set $C \subset \mathbb{R}^n$ is a cone if $\lambda x \in C$ for all $\lambda \geq 0$ and all $x \in C$.

**Definition 2.** A polyhedron of the form $\mathcal{P} = \{x \in \mathbb{R}^n | Ax \geq 0\}$ is called a polyhedral cone.

**Theorem 1.** Let $C \subset \mathbb{R}^n$ be the polyhedral cone defined by the matrix $A$. Then the following are equivalent:

1. The zero vector is an extreme point of $C$.
2. The cone $C$ does not contain a line.
3. The rows of $A$ span $\mathbb{R}^n$. 
Comments on Polyhedral Cones

- Notice that the origin is a member of every polyhedral cone.
- Furthermore, the origin is the only possible extreme point.
- A polyhedral cone that has the origin as an extreme point is called pointed.
- Graphically, a pointed cone looks like what we would ordinarily call a cone.
The Recession Cone

• Consider a nonempty polyhedron \( \mathcal{P} = \{ x \in \mathbb{R}^n | Ax \geq b \} \) and fix a point \( y \in \mathcal{P} \).

• The recession cone at \( y \) is the set of all directions along which we can move indefinitely from \( y \) and still be in \( \mathcal{P} \), i.e.,

\[
\{ d \in \mathbb{R}^n | A(y + \lambda d) \geq b \ \forall \lambda \geq 0 \}.
\]

• This set turns out to be

\[
\{ d \in \mathbb{R}^n | Ad \geq 0 \}
\]

and is hence a polyhedral cone independent of \( y \).

• The nonzero elements of the recession cone are called the rays of \( \mathcal{P} \).

• For a polyhedron in standard form, the rays must satisfy \( Ad = 0, \ d \geq 0 \).
**Extreme Rays**

**Definition 3.**

1. A nonzero element $x$ of a polyhedral cone $C \subseteq \mathbb{R}^n$ is called an extreme ray if there are $n - 1$ linearly independent constraints binding at $x$.

2. An extreme ray of the recession cone associated with a polyhedron $\mathcal{P}$ is also called an extreme ray of $\mathcal{P}$.

- Note that if $d$ is an extreme ray, then so is $\lambda d$ for all $\lambda \geq 0$.
- Two extreme rays are equivalent if one is a multiple of the other.
- When we consider the set of all extreme rays, we will only consider one ray from each equivalence class.
- Note that a polyhedral cone has a finite number of “non-equivalent” extreme rays.
Theorem 2. Consider the problem of minimizing $c^T x$ over a pointed polyhedral cone $C$. The optimal cost is $-\infty$ if and only if some extreme ray $d$ of $C$ satisfies $c^T d < 0$.

Proof:
Characterizing Unbounded LPs

**Theorem 3.** Consider the LP \( \min \{ c^T x \mid Ax \geq b \} \) and assume the feasible region has at least one extreme point. The optimal cost is equal to \(-\infty\) if and only if some extreme ray \( d \) satisfies \( c^T d < 0 \).

**Proof:**
Unboundedness in the Simplex Method

• If we have a standard form problem which is unbounded, the simplex algorithm provides an extreme ray satisfying $c^\top d < 0$.

• When simplex terminates, there is a column $j$ with negative reduced cost and for which basic direction $j$ belongs to the recession cone.

• It is easy to show that this basic direction is an extreme ray of the recession cone.
Representation of Polyhedra

**Theorem 4.** Let \( \mathcal{P} = \{x \in \mathbb{R}^n\} \) be a nonempty polyhedron with at least one extreme point. Let \( x^1, \ldots, x^k \) be the extreme points and \( w^1, \ldots, w^r \) be the extreme rays. Then

\[
P = \left\{ \sum_{i=1}^k \lambda_i x^i + \sum_{j=1}^r \theta_j w^j \mid \lambda_i \geq 0, \theta_j \geq 0, \sum_{i=1}^k \lambda_i = 1 \right\}.
\]

**Proof:**
Corollaries to the Representation Theorem

**Corollary 1.** A nonempty bounded polyhedron, is the convex hull of its extreme points.

**Corollary 2.** A nonempty polyhedron is bounded if and only if it has no extreme rays.

**Corollary 3.** Every element of a polyhedral cone can be expressed as a nonnegative linear combination of extreme rays.
The Converse of the Representation Theorem

Definition 4. A set $Q$ is finitely generated if it is of the form

$$P = \left\{ \sum_{i=1}^{k} \lambda_i x^i + \sum_{j=1}^{r} \theta_j w^j \mid \lambda_i \geq 0, \theta_j \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$ 

for given vectors $x^1, \ldots, x^k$ and $w^1, \ldots, w^r$ in $\mathbb{R}^n$.

Theorem 5. Every finitely generated set is a polyhedron. The convex hull of finitely many vectors is a bounded polyhedron, also called a polytope.