Reading for This Lecture

- Bertsimas 4.4-4.6
More on Complementary Slackness

• Recall the complementary slackness conditions,

\[ p^\top (Ax - b) = 0, \]
\[ (c^\top - p^\top A)x = 0. \]

• If the primal is in standard form, then any feasible primal solution satisfies the first condition.

• If the dual is in standard form, then any feasible dual solution satisfies the second condition.

• Typically, we only need to worry about satisfying the second condition, which is enforced by the simplex method.
Dual Variables and Marginal Costs

• Consider an LP in standard form with a nondegenerate, optimal basic feasible solution $x^*$ and optimal basis $B$.

• Suppose we wish to perturb the right hand side slightly by replacing $b$ with $b + d$.

• As long as $d$ is “small enough,” we have $B^{-1}(b + d) > 0$ and $B$ is still an optimal basis.

• The optimal cost of the perturbed problem is

$$c_B^T B^{-1} (b + d) = p^T (b + d)$$

• This means that the optimal cost changes by $p^T d$.

• Hence, we can interpret the optimal dual prices as the marginal cost of changing the right hand side of the $i^{th}$ equation.
Economic Interpretation

- The dual prices, or *shadow prices* can allow us to put a value on resources.
- Consider the simple product mix problem from Lecture 9.
- By examining the dual variable for the production hours constraint, we can determine the **value of an extra hour of production time**.
- We can also determine the maximum amount we would be willing to pay to borrow extra cash.
- Note that the reduced costs can be thought of as the shadow prices associated with the nonnegativity constraints.
Economic Interpretation of Optimality

- Consider again the product mix example from the Lecture 9.
- Using the shadow prices, we can determine how much each product “costs” in terms of its constituent resources.
- The reduced cost of a product is the difference between its selling price and the (implicit) cost of the constituent resources.
- If we discover a product whose “cost” is less than its selling price, we try to manufacture more of that product to increase profit.
- With the new product mix, the demand for various resources is changed and their prices are adjusted.
- We continue until there is no product with cost less than its selling price.
- This is the same as having the reduced costs nonpositive (recall this was a maximization problem).
- Complementary slackness says that we should only manufacture products for which cost and selling price are equal.
- This can be viewed as a sort of multi-round auction.
Shadow Prices in AMPL

Again, recall the model from Lecture 9.

ampl: model simple.mod
ampl: solve;
CPLEX 7.0.0: optimal solution; objective 105000
2 simplex iterations (0 in phase I)
ampl: display hours;
hours = 0.5

- This tells us that the optimal dual value of the hours constraint is 0.5.
- Increasing the hours by 2000 will increase profit by \((2000)(0.5) = $1000\).
- Hence, we should be willing to pay up to \$.50/hour for additional hours (as long as the solution remains feasible).
The Dual Simplex Method

• We now present a **dual version** of the simplex method in tableau form.

• Recall the simplex tableau

<table>
<thead>
<tr>
<th>$-c_B^T x_B$</th>
<th>$\bar{c}_1$</th>
<th>$\cdots$</th>
<th>$\bar{c}_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_B(1)$</td>
<td>$B^{-1}A_1$</td>
<td>$\cdots$</td>
<td>$B^{-1}A_n$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_B(m)$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

• In the dual simplex method, the basic variables are allowed to take on **negative values**, but we keep the reduced costs nonnegative.
Choosing the Pivot Element

• The pivot row is any row in which the value of the basic variable is negative.

• To determine the pivot column, we perform a ratio test.

• The ratio test determines the largest step length that will maintain dual feasibility, i.e., keep the reduced costs nonnegative.

• Consider the pivot row $v$—if $v_i \geq 0 \ \forall i$, then the optimal dual cost is $+\infty$ (the primal problem is infeasible).

• Otherwise, if $v_i < 0$, compute the ratio $-\frac{\bar{c}_i}{v_i}$.

• The pivot column is one of the columns with the minimum ratio.

• Pivoting is done in exactly the same way as before.
Comments on Dual Simplex

• Note that a given basis determines both a unique solution to the primal and a unique solution to the dual.

\[
\begin{align*}
x_B &= B^{-1}b \\
p^\top &= c_B B^{-1}
\end{align*}
\]

• Both the primal and dual solutions are basic and either one, or both, may be feasible.

• If they are both feasible, then they are both optimal.

• Both versions of the simplex method go from one adjacent basic solution to another until reaching optimality.

• Both versions either terminate in a finite number of steps or cycle.

• The dual simplex method is not exactly the same as the simplex method applied to the dual.
Why Use Dual Simplex

- Note that when we can’t find a primal feasible basis, we may be able to find a dual feasible basis.

- For a **primal problem in standard form** with nonnegative costs, we always have a **dual feasible solution**.

- Suppose we have an optimal basis and we **change the right hand side** so that the basis becomes primal infeasible.

- The **basis will still be dual feasible** and so we can continue on with the dual simplex method.

- Note that we can **switch back and forth** between the two methods.
Dual Degeneracy

• Consider an LP in standard form.
• Recall that the reduced costs are the slack in the dual constraints.
• The reduced costs that are zero correspond to binding dual constraints.
• A dual solution is degenerate if and only if the reduced cost of some nonbasic variable is zero.
• Primal and dual degeneracy are not connected—two bases can lead to the same primal solution, but different dual solutions and vice versa.
• Two bases can even lead to the same primal solution and different dual solutions, one of which is feasible and the other of which is not.
• Dual degeneracy can also cause problems.
Geometric Interpretation of Optimality

• Suppose we have a problem in inequality form, so that the dual is in standard form, and a basis $B$.

• If $I$ is the index set of binding constraints at the corresponding (nondegenerate) BFS, and we enforce complementary slackness, then dual feasibility is equivalent to

$$\sum_{i \in I} p_i a_i = c.$$

• In other words, the objective function must be a nonnegative combination of the binding constraints.

• We can easily picture this graphically.
**Farkas’ Lemma**

**Proposition 1.** Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ be given. Then exactly one of the following holds:

1. $\exists x \geq 0$ such that $Ax = b$.

2. $\exists p$ such that $p^\top A \succeq 0^\top$ and $p^\top b < 0$.

- This is closely related to the geometric interpretation of optimality just discussed.
- There are many equivalent version of Farkas’ Lemma from which we can derive optimality conditions.
- Note that when the dual simplex algorithm stops because of infeasibility, then the pivot row provides a proof.
An Asset Pricing Model

- Suppose we are in a market that operates for one period and in which \( n \) different assets are traded.
- At the end of the period, the market can be in \( m \) different possible states.
- Each asset \( i \) has a given price \( p_i \) at the beginning of the period.
- We have a payoff matrix \( R \) which determines the price \( r_{si} \) of asset \( i \) at the end of the period if the market is in state \( s \).
- Note that we are allowed to sell short, which means selling some quantity of asset \( i \) at the beginning of the period and buying it back at the end.
- Asset pricing models typically try to determine prices for which there are no arbitrage opportunities.
- This means there is no portfolio with a negative cost, but a positive return in every state.
Applying Linear Programming

• We can develop a linear program to look for arbitrage opportunities.

• Suppose we let the vector $x$ represent our portfolio at the beginning of the period.

• The condition that our return should be positive in every state is simply

$$Rx \geq 0$$

• The condition that the portfolio has negative cost is simply

$$p^\top x \leq 0$$

• Hence, we can simply solve the LP $\min\{p^\top x | Rx \geq 0\}$. 
Asset Pricing Using Farkas’ Lemma

- The absence of arbitrage is equivalent to the condition that $Rx \geq 0 \Rightarrow p^\top x \geq 0$.

- This is the same as the LP above have a nonnegative optimal solution.

- By Farkas’ Lemma, the absence of arbitrage opportunities is equivalent to the existence of a vector of nonnegative state prices $q$ such that

$$p = q^\top R$$

- Hence, if we determine such state prices and use them to value existing assets, we eliminate the possibility of arbitrage.

- This is a key concept in modern finance theory.