Reading for This Lecture

- Bertsimas 2.5-2.7
Existence of Extreme Points

Definition 1. A polyhedron $\mathcal{P} \in \mathbb{R}^n$ contains a line if there exists a vector $x \in \mathcal{P}$ and a nonzero vector $d \in \mathbb{R}^n$ such that $x + \lambda d \in \mathcal{P}$ $\forall \lambda \in \mathbb{R}$.

Theorem 1. Suppose that the polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n | Ax \geq b\}$ is nonempty. Then the following are equivalent:

- The polyhedron $\mathcal{P}$ has at least one extreme point.
- The polyhedron $\mathcal{P}$ does not contain a line.
- There exist $n$ rows of $A$ that are linearly independent.
Optimality of Extreme Points

**Theorem 2.** Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polyhedron and consider the problem $\min_{x \in \mathcal{P}} c^T x$ for a given $c \in \mathbb{R}^n$. If $\mathcal{P}$ has at least one extreme point and there exists an optimal solution, then there exists an optimal solution that is an extreme point.

**Proof:**
Optimality in Linear Programming

• For linear optimization, a finite optimal cost is equivalent to the existence of an optimal solution.

• The previous result can be strengthened.

• Since any linear programming problem can be written in standard form, we can derive the following result:

Theorem 3. Consider the linear programming problem of minimizing $c^T x$ over a nonempty polyhedron. Then, either the optimal cost is $-\infty$ or there exists an optimal solution which is an extreme point.
Representation of Polyhedra

**Theorem 4.** A nonempty, bounded polyhedron is the convex hull of its extreme points.

**Theorem 5.** The convex hull of a finite set of vectors is a polyhedron.

**Notes:**
Example: Product Mix

• In this example, we consider Top Brass Trophy, a shop that manufactures two kinds of trophies, football and soccer.

• Resource requirements
  – Football trophies: 1 brass football, 1 plaque, 4 board feet of wood.
  – Soccer Trophies: 1 brass soccer ball, 1 plaque, 2 board feet of wood.

• Resource constraints
  – 1000 footballs
  – 1500 soccer balls
  – 1750 plaques
  – 4800 board feet of wood

• Profit is $12 on football trophies and $9 on soccer trophies.

• The goal is to maximize profit.
Top Brass Example: Formulation

• What are the decision variables?
• What is the objective function?
• What are the constraints?
Top Brass Example: Solving

- Basic scheme
  - Rewrite constraints in standard form.
  - Find an initial basic feasible solution.
  - Move to an adjacent vertex that improves the solution value.
  - Keep moving until no further improvement is possible.

- **Question**: What sets of variables do not form a basis?
Top Brass Example: Degeneracy

- Suppose we had the additional constraint $3x_1 + 2x_2 \leq 5000$.
- Note that this constraint is *redundant*.
- This new constraint is *linearly dependent* on the other constraints.
- Initially, we may not know this.
- What could happen to our solution method?
Top Brass Example: Alternative Representation

- Denote the polyhedron from the example by $\mathcal{P}$.
- Denote the extreme points of $\mathcal{P}$ by $p_1, \ldots, p_6$.
- Then we could also represent $\mathcal{P}$ as

$$\mathcal{P} = \{ x \in \mathbb{R}^n : x = \sum_{i=1}^{6} \lambda_ip_i, \sum_{i=1}^{6} \lambda_i = 1, \lambda_i \geq 0, i = 1, \ldots, 6 \}$$

- Rewriting the objective function as

$$\max c^T \sum_{i=1}^{6} \lambda_ip_i,$$

we have another form of the LP.
What We’ve Learned So Far

- We are interested in the extreme points of polyhedra.
- There is a one-to-one correspondence between the extreme points of a polyhedron and the basic feasible solutions.
- We can construct basic solutions by
  - Choosing a basis $B$ of $m$ linearly independent columns of $A$.
  - Solve the system $Bx_B = b$ to obtain the values of the basic variables.
  - Set $x_N = 0$.
- We can move between adjacent (nondegenerate) basic solutions by removing one column of the basis and replacing it with another.
- In the presence of degeneracy, we might stay at the same extreme point.
- These are the building blocks we need to construct algorithms for solving LPs.