Reading for This Lecture

- Operations Research Methods and Models
- Bertsimas 1.1-1.2, 1.4-1.5, 1.3 (optional)
Review from Last Time

• Recall that a mathematical model consists of:
  – Decision variables
  – Constraints
  – Objective Function
  – Parameters and Data

The general form of a mathematical programming model is:

\[
\begin{align*}
\min \text{ or } \max \ f(x_1, \ldots, x_n) \\
\text{s.t. } g_i(x_1, \ldots, x_n) \begin{cases} \leq \\ = \\ \geq \end{cases} b_i
\end{align*}
\]

We might also want the values of the variables to belong to discrete set \( X \).
Example of a Mathematical Program: The Diet Problem

- **Goal**: Choose the cheapest menu satisfying nutritional requirements.
- What is the input data?
- What is the formulation in words?
Critique of the Model

• What are the possible problems with this model?
A Little History

• **George Dantzig** is considered to be the father of linear programming.

• The **diet problem** was one of the first applications of linear programming.

• It took **120 man-days** to solve a problem with 9 constraints and 77 variables by hand!

• Later, Dantzig tried to lose weight by designing his own diet.
  – The first solution he came up with contained several gallons of **vinegar**.
  – After deleting vinegar from the list of foods, the new solution contained approximately **200 bouillon cubes**.
  – This illustrates one of the potential hazards of math modeling.
Representing Math Models: Vectors and Matrices

• An \( m \times n \) matrix is an array of \( mn \) real numbers:

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]

• \( A \) is said to have \( n \) columns and \( m \) rows.

• An \( n \)-dimensional column vector is a matrix with one column.

• An \( n \)-dimensional row vector is a matrix with one row.

• By default, a vector will be considered a column vector.

• The set of all \( n \)-dimensional vectors will be denoted \( \mathbb{R}^n \).

• The set of all \( m \times n \) matrices will be denoted \( \mathbb{R}^{m \times n} \).
Matrices

• The *transpose* of a matrix $A$ is

\[
A^T = \begin{bmatrix}
    a_{11} & a_{21} & \cdots & a_{m1} \\
    a_{12} & a_{22} & \cdots & a_{m2} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{1n} & a_{2n} & \cdots & a_{mn}
\end{bmatrix}
\]

• If $x, y \in \mathbb{R}^n$, then $x^T y = \sum_{i=1}^{n} x_i y_i$.

• This is called the *inner product* of $x$ and $y$.

• If $A \in \mathbb{R}^{m \times n}$, then $A_j$ is the $j^{th}$ column, and $a_j$ is the $j^{th}$ row.

• If $A \in \mathbb{R}^{m \times k}$, $B \in \mathbb{R}^{k \times n}$, then $[AB]_{ij} = a_i^T B_j$. 
Linear Functions

• A linear function $f : \mathbb{R}^n \to \mathbb{R}$ is a weighted sum, written as

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} c_i x_i$$

for given coefficients $c_1, \ldots, c_n$.

• We can write $x_1, \ldots, x_n$ and $c_1, \ldots, c_n$ as vectors $x, c \in \mathbb{R}^n$ to obtain:

$$f(x) = c^T x$$

• In this way, a linear function can be represented simply as a vector.

• We will consider only models defined by linear functions.
Back to the Diet Problem

• How do we write the diet problem mathematically?
• What are the decision variables?
• What is the objective function?
• What are the constraints?
Putting It All Together

- Using matrix notation, we can write our current formulation as

$$\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad l \leq Ax \leq u
\end{align*}$$

- What are we missing?
Linear Programs

- What we have just seen is an example of a linear program.

- In general, we can write a linear program as

\[
\begin{align*}
\text{minimize} \quad & c^T x \\
\text{s.t.} \quad & a_i^T x \geq b_i \quad \forall i \in M_1 \\
\quad & a_i^T x \leq b_i \quad \forall i \in M_2 \\
\quad & a_i^T x = b_i \quad \forall i \in M_3 \\
\quad & x_j \geq 0 \quad \forall j \in N_1 \\
\quad & x_j \leq 0 \quad \forall j \in N_2 
\end{align*}
\]

- This in turn can be written equivalently as

\[
\begin{align*}
\text{minimize} \quad & c^T x \\
\text{s.t.} \quad & Ax \geq b
\end{align*}
\]

- How do we do this?
Standard Form

- To solve a linear program, we must put it in the following standard form:

\[
\begin{align*}
\min c^T x \\
\text{s.t. } Ax &= b \\
x &\geq 0
\end{align*}
\]

- How do we do this?
Two Crude Petroleum Example

- Two Crude Petroleum distills crude from two sources:
  - Saudi Arabia
  - Venezuela

- They have three main products:
  - Gasoline
  - Jet fuel
  - Lubricants

- Yields

<table>
<thead>
<tr>
<th></th>
<th>Gasoline</th>
<th>Jet fuel</th>
<th>Lubricants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Saudi Arabia</td>
<td>0.3 barrels</td>
<td>0.4 barrels</td>
<td>0.2 barrels</td>
</tr>
<tr>
<td>Venezuela</td>
<td>0.4 barrels</td>
<td>0.2 barrels</td>
<td>0.3 barrels</td>
</tr>
</tbody>
</table>
Two Crude Petroleum Example (cont.)

• Availability and cost

<table>
<thead>
<tr>
<th></th>
<th>Availability</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>Saudi Arabia</td>
<td>9000 barrels</td>
<td>$20/barrel</td>
</tr>
<tr>
<td>Venezuela</td>
<td>6000 barrels</td>
<td>$15/barrel</td>
</tr>
</tbody>
</table>

• Production Requirements (per day)

<table>
<thead>
<tr>
<th></th>
<th>Gasoline</th>
<th>Jet fuel</th>
<th>Lubricants</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2000 barrels</td>
<td>1500 barrels</td>
<td>500 barrels</td>
</tr>
</tbody>
</table>

• **Objective**: Minimize production cost.
Modeling the Two Crude Production Problem

• What are the decision variables?
• What is the objective function?
• What are the constraints?
Linear Programming Formulation of Two Crude Example

• This yields the following LP formulation:

\[
\begin{align*}
\text{min } & \quad 20x_1 + 15x_2 \\
\text{s.t. } & \quad 0.3x_1 + 0.4x_2 \geq 2.0 \\
& \quad 0.4x_1 + 0.2x_2 \geq 1.5 \\
& \quad 0.2x_1 + 0.3x_2 \geq 0.5 \\
& \quad 0 \leq x_1 \leq 9 \\
& \quad 0 \leq x_2 \leq 6
\end{align*}
\]

• How can we solve this problem?

• What are the possible outcomes of solving such a problem?