

Advanced Operations Research Techniques

IE316

Lecture 15

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Reading for This Lecture

- Bertsimas Chapter 5

Global Dependence on the Right-hand Side Vector

- Consider a family of polyhedra parameterized by the vector b

$$\mathcal{P}(b) = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$$

- Note that

$$S = \{b : \mathcal{P}(b) \text{ is nonempty}\} = \{Ax : x \geq 0\}$$

is a convex set.

- We now consider the function $F(b) = \min_{x \in \mathcal{P}(b)} c^T x$.
- In what follows, we will assume feasibility of the dual and hence that $F(b)$ is finite for all $b \in S$.
- We will try to characterize the function $F(b)$.

Characterizing $F(b)$

- For a particular vector \hat{b} , suppose there is a **nondenerate optimal basic feasible solution** given by basis B .
- As before, nondegeneracy implies that we can perturb \hat{b} without changing the optimal basis.
- Therefore, we have

$$F(b) = c_B^T B^{-1} b = p^T b, \quad \text{for } b \text{ “close to” } \hat{b}.$$

- This means that in the vicinity of \hat{b} , $F(b)$ is a **linear function of b** .
- Consider the extreme points p^1, \dots, p^N of the dual polyhedron.
- There must be an extremal optimum to the dual and so we can rewrite $F(b)$ as

$$F(b) = \max_{i=1, \dots, N} (p^i)^T b$$

- Hence, $F(b)$ is a **piecewise linear convex function**.

Another Parameterization

- Now consider the function $f(\theta) = F(\hat{b} + \theta d)$ for a particular vector \hat{b} and direction d .
- Using the same approach, we obtain

$$f(\theta) = \max_{i=1, \dots, N} (p^i)^T (\hat{b} + \theta d), \quad \hat{b} + \theta d \in S.$$

- Again, this is a **piecewise linear convex function**.

The Set of all Dual Optimal Solutions

Consider once more the function $F(b)$.

Definition 1. A vector $p \in \mathbb{R}^m$ is a **subgradient** of F at \hat{b} if $F(\hat{b}) + p^T(b - \hat{b}) \leq F(b)$.

Theorem 1. Suppose that the linear program $\min\{c^T x : Ax = \hat{b}, x \geq 0\}$ is feasible and that the optimal cost is finite. Then p is an optimal solution to the dual if and only if it is a subgradient of F at \hat{b} .

Global Dependence on the Cost Vector

- We have similar results for **dependence on the cost vector**.
- As before, assume primal feasibility and define the dual feasible set by

$$Q(c) = \{p : p^T A \leq c\}$$

and set $T = \{c : Q(c) \text{ is nonempty}\}$.

- It is easy to show that T is again a convex set.
- Defining a function $G(c)$ to represent the optimal primal cost corresponding to vector c , we see as before that

$$G(c) = \min_{i=1, \dots, N} c^T x^i$$

where x^1, \dots, x^N are the extreme points of the primal polyhedron.

Characterizing $G(c)$

- As before, $G(c)$ is a **piecewise linear concave function**.
- Furthermore, if for some cost vector \hat{c} , there is unique optimal solution x^* , then $G(c)$ is linear in the vicinity of \hat{c} .
- The set of all optimal solutions is again given by the **subgradients of $G(c)$** .

Parametric Programming

- For a fixed matrix A , vectors b and c , and direction d and consider the problem

$$\begin{aligned} \min & (c + \theta d)^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{aligned}$$

- If $g(\theta)$ is the optimal cost for a given θ , then as before

$$g(\theta) = \min_{i=1, \dots, N} (c + \theta d)^T x^i$$

where x^1, \dots, x^N are the extreme points of the feasible set.

- Hence, $g(\theta)$ is piecewise linear and concave.
- We now derive a systematic method for obtaining $g(\theta)$ for all θ .

The Basic Idea

- For a fixed basis B and a fixed θ , the reduced costs are linear functions of θ .
- Hence, we can easily determine the range of values of θ for which the current basis is optimal.
- The ends of this interval determine the nearest breakpoints of $g(\theta)$.
- By changing θ to the next breakpoint and performing a change of basis, we obtain a new interval and a new breakpoint.
- This process can be continued to obtain all breakpoints.

The Parametric Simplex Method

- Determine an **initial feasible basis**.
- Determine the interval $[\theta_1, \theta_2]$ for which this basis is **optimal**.
- Determine a variable j whose reduced cost is **nonpositive** for $\theta \geq \theta_2$.
- If the corresponding column has no positive entries, then the problem is **unbounded** for $\theta > \theta_2$.
- Otherwise, rotate column j into the basis.
- Determine a new interval $[\theta_2, \theta_3]$ in which the current basis is optimal.
- **Iterate** to find all breakpoint $\geq \theta_1$.
- Repeat the process to find breakpoints $\leq \theta_1$.

Comments on Parametric Simplex Method

- Note that as long as each interval has a positive length, **this process will terminate finitely**.
- If some interval has zero length, **cycling is possible**.
- We can prevent cycling with anticycling rules as in the regular simplex algorithm.
- We can perform the same analysis for a parametrically defined right hand side using **dual simplex**.

AMPL: Displaying Auxiliary Values with Suffixes

- In **AMPL**, it's possible to display many of the auxiliary values we've been looking at using **suffixes**.
- For example, to display the **reduced cost** of a variable, type the variable name with the suffix **.rc**.
- Using the product mix example (**prod.mod** and **prod1.dat**),

```
AMPL: display make;
make [*] :=
gadgets 15000
widgets 20000
;
```

```
AMPL: display make.rc;
make.rc [*] :=
gadgets 0
widgets 0
;
```

AMPL: Other Auxiliary Information

- You can display the *status* of each variable

```
ampl: display make.sstatus;  
make.sstatus [*] :=  
gadgets  bas  
widgets  bas
```

- You can also display such things as the [slack in the constraints](#)

```
ampl: display hours_limit.slack;  
hours_limit.slack = 10000
```

- Or the [status of a slack variable](#)

```
ampl: display hours_limit.status;  
hours_limit.status = bas
```

- A list of all the possible suffixes is on the [AMPL Web site](#).

AMPL: Sensitivity

- AMPL does not have built-in **sensitivity analysis** commands.
- AMPL/CPLEX does provide such capability.
- To get sensitivity information, type the following

```
ampl: option cplex_options 'sensitivity';
```

- Solve the model from product mix model:

```
ampl: solve;
```

```
CPLEX 7.0.0: sensitivity
```

```
CPLEX 7.0.0: optimal solution; objective 105000
```

```
2 simplex iterations (0 in phase I)
```

```
suffix up OUT;
```

```
suffix down OUT;
```

```
suffix current OUT;
```

AMPL: Accessing Sensitivity Information

Access sensitivity information using the suffixes *.up* and *.down*.

```
ampl: display hours_limit.up;  
hours_limit.up = 180000
```

```
ampl: let max_prd := 190000;  
ampl: solve;  
CPLEX 7.0.0: sensitivity
```

```
CPLEX 7.0.0: optimal solution; objective 135000  
1 simplex iterations (1 in phase I)  
ampl: display make;  
make [*] :=  
gadgets 45000  
widgets 0  
;
```

AMPL: Accessing Sensitivity Information (cont.)

```
ampl: display make.rc;  
make.rc [*] :=  
gadgets    0  
widgets   -1.5  
;
```

```
ampl: display make.sstatus;  
make.sstatus [*] :=  
gadgets   bas  
widgets   low  
;
```

A Case Study in Sensitivity Analysis

- This study is from Section 1.2 and 5.1 of the book.
- The following new computer models are being introduced by **DEC**.

System	Price	# disk drives	# 256K boards
GP-1	\$60,000	0.3	4
GP-2	\$40,000	1.7	2
GP-3	\$30,000	0	2
WS-1	\$30,000	1.4	2
WS-2	\$15,000	0	1

Scenario

- The following difficulties were anticipated:
 - The in-house CPU supply is limited to 7,000 units.
 - The supply of disk drives is uncertain (3,000-7,000).
 - The supply of memory boards is also limited (8,000-16,000).
- Some maximum and minimum demand information is known.
- Possible Approaches
 - To address the disk drive shortage, GP-1 systems could be produced with **no disk drives**.
 - In addition, the GP-1 systems could be built using two **alternative** memory boards instead of four standard boards.
 - 4,000 alternative boards are available.

Questions to Be Answered

- Should the GP-1s be manufactured **with or without disk drives**?
- Should the **alternative memory boards** be used?
- The small staff can only devote their efforts to either finding an alternative source of disk drives or memory boards. Which one?
- We can answer these questions using **sensitivity analysis**.

Analysis

Alt. Boards	Drives	Revenue	GP-1	GP-2	GP-3	WS-1	WS-2
no	no	145	0	2.5	0	0.5	2
yes	no	248	1.8	2	0	1	2
no	yes	133	0.272	1.304	0.3	0.5	2.7
yes	yes	213	1.8	1.035	0.3	0.5	2.7

Mode	No Disk Drives	Disk Drives
Revenue	248	213
Shadow price for boards	15	0
Range for boards	$[-1.5, 2]$	$[-1.62, \infty]$
Shadow price for drives	0	23.52
Range for drives	$[-0.2, 0.75]$	$[-0.91, 1.13]$