Reading for This Lecture

- Bertsimas 4.4-4.6
More on Complementary Slackness

- Recall the **complementary slackness** conditions,

\[
\begin{align*}
p^T (Ax - b) &= 0, \\
(c^T - p^T A)x &= 0.
\end{align*}
\]

- If the primal is in standard form, then any feasible primal solution satisfies the first condition.

- If the dual is in standard form, then any feasible dual solution satisfies the second condition.

- Typically, we only need to worry about satisfying the second condition, which is enforced by the simplex method.
Dual Variables and Marginal Costs

- Consider an LP in standard form with a nondegenerate, optimal basic feasible solution $x^*$ and optimal basis $B$.
- Suppose we wish to perturb the right hand side slightly by replacing $b$ with $b + d$.
- As long as $d$ is “small enough,” we have $B^{-1}(b + d) > 0$ and $B$ is still an optimal basis.
- The optimal cost of the perturbed problem is

$$c_B^T B^{-1}(b + d) = p^T (b + d)$$

- This means that the optimal cost changes by $p^T d$.
- Hence, we can interpret the optimal dual prices as the marginal cost of changing the right hand side of the $i^{th}$ equation.
Economic Interpretation

- The dual prices, or *shadow prices* can allow us to put a value on resources.
- Consider the simple product mix problem from the Lecture 10.
- By examining the dual variable for the production hours constraint, we can determine *the value of an extra hour of production time*.
- We can also determine the maximum amount we would be willing to pay to borrow extra cash.
- Note that the reduced costs are the shadow prices associated with the nonnegativity constraints.
Economic Interpretation of Optimality

- Consider again the product mix example from the Lecture 9.
- Using the shadow prices, we can determine how much each product “costs” in terms of its constituent resources.
- The reduced cost of a product is the difference between its selling price and the (implicit) cost of the constituent resources.
- If we discover a product whose “cost” is less than its selling price, we try to manufacture more of that product to increase profit.
- With the new product mix, the demand for various resources is changed and their prices are adjusted.
- We continue until there is no product with cost less than its selling price.
- This is the same as having the reduced costs nonnegative.
- Complementary slackness says that we should only manufacture products for which cost and selling price are equal.
- This can be viewed as a sort of multi-round auction.
**Shadow Prices in AMPL**

Again, recall the model from the Lecture 10.

ampl: model simple.mod
ampl: solve;
CPLEX 7.0.0: optimal solution; objective 105000
2 simplex iterations (0 in phase I)
ampl: display hours;
hours = 0.5

- This tells us that the **optimal dual value** of the hours constraint is 0.5.
- Increasing the hours by 2000 will increase profit by $(2000)(0.5) = \$1000$.
- Hence, we should be willing to pay up to $.50/\text{hour}$ for additional hours (as long as the solution remains feasible).
The Dual Simplex Method

- We now present a **dual version** of the simplex method in tableau form.
- Recall the simplex tableau

\[
\begin{array}{cccc}
- c_B^T x_B & \bar{c}_1 & \cdots & \bar{c}_n \\
 x_B(1) & B^{-1} A_1 & \cdots & B^{-1} A_n \\
 \vdots & & & \\
x_B(m) & & & \\
\end{array}
\]

- In the dual simplex method, **the basic variables are allowed to take on negative values**, but we keep the reduced costs nonnegative.
Choosing the Pivot Element

- The **pivot row** is any row in which the value of the basic variable is negative.

- To determine the **pivot column**, we perform a **ratio test**.

- The ratio test determines the largest step length that will **maintain dual feasibility**, i.e., keep the reduced costs nonnegative.

- Consider the pivot row \( v \)—if \( v_i \geq 0 \forall i \), then the optimal dual cost is \( +\infty \) (the primal problem is infeasible).

- Otherwise, if \( v_i < 0 \), compute the ratio \( -\frac{\bar{c}_i}{v_i} \).

- The pivot column is one of the columns with the **minimum ratio**.

- Pivoting is done in exactly the same way as before.
Comments on Dual Simplex

• Note that a given basis determines both a unique solution to the primal and a unique solution to the dual.

\[ x_B = B^{-1}b \]
\[ p^T = c_B B^{-1} \]

• Both the primal and dual solutions are basic and either one, or both, may be feasible.

• If they are both feasible, then they are both optimal.

• Both versions of the simplex method go from one adjacent basic solution to another until reaching optimality.

• Both versions either terminate in a finite number of steps or cycle.

• The dual simplex method is not exactly the same as the simplex method applied to the dual.
Why Use Dual Simplex

- Note that when we can’t find a primal feasible basis, we may be able to find a dual feasible basis.

- For a primal problem in standard form with nonnegative costs, we always have a dual feasible solution.

- Suppose we have an optimal basis and we change the right hand side so that the basis becomes primal infeasible.

- The basis will still be dual feasible and so we can continue on with the dual simplex method.

- Note that we can switch back and forth between the two methods.
Dual Degeneracy

• Consider an LP in standard form.
• Recall that the reduced costs are the slack in the dual constraints.
• The reduced costs that are zero correspond to binding dual constraints.
• A dual solution is degenerate if and only if the reduced cost of some nonbasic variable is zero.
• Primal and dual degeneracy are not connected—two bases can lead to the same primal solution, but different dual solutions and vice versa.
• Two bases can even lead to the same primal solution and different dual solutions, one of which is feasible and the other of which is not.
• Dual degeneracy can also cause problems.
Geometric Interpretation of Optimality

• Suppose we have a problem in inequality form, so that the dual is in standard form, and a basis $B$.

• If $I$ is the index set of binding constraints at the corresponding (nondegenerate) BFS, and we enforce complementary slackness, then dual feasibility is equivalent to

$$\sum_{i \in I} p_i a_i = c.$$ 

• In other words, the objective function must be a nonnegative combination of the binding constraints.

• We can easily picture this graphically.
Farkas’ Lemma

Proposition 1. Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ be given. Then exactly one of the following holds:

1. $\exists x \geq 0$ such that $Ax = b$.
2. $\exists p$ such that $p^T A \geq 0^T$ and $p^T b < 0$.

- This is closely related to the geometric interpretation of optimality just discussed.
- There are many equivalent versions of Farkas’ Lemma from which we can derive optimality conditions.
- Note that when the dual simplex algorithm stops because of infeasibility, then the pivot row provides a proof.
An Asset Pricing Model

• Suppose we are in a market that operates for one period and in which $n$ different assets are traded.

• At the end of the period, the market can be in $m$ different possible states.

• Each asset $i$ has a given price $p_i$ at the beginning of the period.

• We have a payoff matrix $R$ which determines the price $r_{si}$ of asset $i$ at the end of the period if the market is in state $s$.

• Note that we are allowed to sell short, which means selling some quantity of asset $i$ at the beginning of the period and buying it back at the end.

• Asset pricing models typically try to determine prices for which there are no arbitrage opportunities.

• This means there is no portfolio with a negative cost, but a positive return in every state.
Applying Linear Programming

• We can develop a linear program to look for arbitrage opportunities.
• Suppose we let the vector $x$ represent our portfolio at the beginning of the period.
• The condition that our return should be positive in every state is simply

$$Rx \geq 0$$

• The condition that the portfolio has negative cost is simply

$$p^T x \geq 0$$

• Hence, we can simply solve the LP $\min\{p^T x | Rx \geq 0\}$. 
Asset Pricing Using Farkas’ Lemma

• The absence of arbitrage is equivalent to the condition that \( Rx \geq 0 \Rightarrow p^T x \geq 0 \).

• This is the same as the LP above have a nonnegative optimal solution.

• By Farkas’ Lemma, the absence of arbitrage opportunities is equivalent to the existence of a vector of nonnegative state prices \( q \) such that

\[
p = q^T R
\]

• Hence, if we determine such state prices and use them to value existing assets, we eliminate the possibility of arbitrage.

• This is a key concept in modern finance theory.