Reading for This Lecture

• Bertsimas 4.1-4.3
Duality Theory: Motivation

• Consider the following minimization problem

\[
\begin{align*}
\text{minimize} & \quad x^2 + y^2 \\
\text{s.t.} & \quad x + y = 1
\end{align*}
\]

• How could we solve this problem?

• Idea: Consider the function

\[
L(x, y, p) = x^2 + y^2 + p(1 - x - y)
\]

• What can we do with this?
Lagrange Multipliers

• The idea is not to strictly enforce the constraints.
• We associated a Lagrange multiplier, or price, with each constraint.
• Then we allow the constraint to be violated for a price.
• Consider an LP in standard form.
• Using Lagrange multipliers, we can formulate an alternative LP:

\[
\begin{align*}
\text{minimize} & \quad c^T x + p^T (b - Ax) \\
\text{s.t.} & \quad x \geq 0
\end{align*}
\]

• How does the optimal solution of this compare to the original optimum?
Lagrange Multipliers

• Because we haven’t changed the cost of feasible solutions to the original problem, this new problem gives a lower bound.

\[ g(p) = \min_{x \geq 0} \left[ c^T x + p^T (b - Ax) \right] \leq c^T x^* + p^T (b - Ax^*) = c^T x^* \]

• Since each value of \( p \) gives a lower bound, we consider maximizing \( g(p) \).

• Think of this as finding the best lower bound.

• This is known as the dual problem.
Simplifying

• In linear programming, we can obtain an explicit form for the dual.

\[
g(p) = \min_{x \geq 0} \left[ c^T x + p^T (b - Ax) \right] \\
= p^T b + \min_{x \geq 0} (c^T - p^T A)x
\]

• Note that

\[
\min_{x \geq 0} (c^T - p^T A)x = \begin{cases} 
0, & \text{if } c^T - p^T A \geq 0^T, \\
-\infty, & \text{otherwise},
\end{cases}
\]

• Hence, we can show that the dual is equivalent to

\[
\begin{align*}
& \text{maximize} & & p^T b \\
& \text{s.t.} & & p^T A \leq c^T
\end{align*}
\]
Inequality Form

• Suppose we have an LP in inequality form.
• We can add slack variables and convert to standard form with constraints

\[
[A | -I] \begin{bmatrix} x \\ s \end{bmatrix} = b
\]

• This leads to dual constraints

\[
p^T [A | -I] \leq [c^T | 0^T]
\]

• Hence, we get the dual

\[
\begin{align*}
\text{maximize} & \quad p^T b \\
\text{s.t.} & \quad p^T A \leq c^T \\
& \quad p \geq 0
\end{align*}
\]
From the Primal to the Dual

We can dualize general LPs as follows

<table>
<thead>
<tr>
<th>PRIMAL</th>
<th>minimize</th>
<th>maximize</th>
<th>DUAL</th>
</tr>
</thead>
</table>
| constraints | \( \geq b_i \) \ 
\( \leq b_i \) \ 
\( = b_i \) | \( \geq 0 \) \ 
\( \leq 0 \) \ 
free | variables |
| variables | \( \geq 0 \) \ 
\( \leq 0 \) \ 
free | \( \leq c_j \) \ 
\( \geq c_j \) \ 
\( = c_j \) | constraints |
Properties of the Dual

• All equivalent forms of the primal give equivalent forms of the dual.

• The dual of the dual is the primal.

• **Weak Duality:** If \( x \) is a feasible solution to the primal and \( p \) is a feasible solution to the dual, then

\[
p^T b \leq c^T x
\]

• **Corollaries:**
  - If the optimal cost of the primal is \(-\infty\), then the dual is infeasible.
  - If the optimal cost of the dual is \(+\infty\), then the primal is infeasible.
  - If \( x \) is a feasible primal solution and \( p \) is a feasible dual solution such that \( c^T x = p^T b \), then both \( x \) and \( p \) are optimal.
## Relationship of the Primal and the Dual

The following are the possible relationships between the primal and the dual:

<table>
<thead>
<tr>
<th></th>
<th>Finite Optimum</th>
<th>Unbounded</th>
<th>Infeasible</th>
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</thead>
<tbody>
<tr>
<td><strong>Finite Optimum</strong></td>
<td>Possible</td>
<td>Impossible</td>
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Strong Duality

Proposition 1. (Strong Duality) If a linear programming problem has an optimal solution, so does its dual, and the respective optimal costs are equal.

Proof:
More About the Dual

• When we interpret the quantity $c_B B^{-1}$ as the vector of dual prices, the reduced costs are then the slack in the constraints of the dual.

• The condition that all the reduced costs be nonnegative is then equivalent to dual feasibility.

• Hence, the simplex algorithm can be interpreted as maintaining primal feasibility while trying to achieve dual feasibility.

• We will shortly see an alternative algorithm which maintains dual feasibility while trying to achieve primal feasibility.
Complementary Slackness

**Proposition 2.** If $x$ and $p$ are feasible primal and dual solutions, then $x$ and $p$ are optimal if and only if

\[
p^T(Ax - b) = 0, \]
\[
(c^T - p^T A)x = 0.
\]

**Proof:**
Optimality Without Simplex

Let’s consider an LP in standard form. We have now shown that the optimality conditions for (nondegenerate) $x$ are

1. $Ax = b$ (primal feasibility)
2. $x \geq 0$ (primal feasibility)
3. $x_i = 0$ if $p^T a_i < c_i$ (complementary slackness)
4. $p^T A \leq c$ (dual feasibility)

- In standard form, the complementary slackness condition is simply $x^T \bar{c} = 0$.
- This condition is always satisfied during the simplex algorithm, since the reduced costs of the basic variables are zero.