Computational Integer Programming

Lecture 2: Modeling and Formulation

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Reading for This Lecture

- N&W Sections I.1.1-I.1.6
- Wolsey Chapter 1
- CCZ Chapter 2

Formulations and Models

- Our description in the last lecture boiled the modeling process down to two basic steps.
  1. Create a *conceptual model* of the real-world problem.
  2. Translate the conceptual model into a *formulation*.
- In the *conceptual model*, we initially describe what values of the variables we would like to allow in logical/conceptual terms (the feasible set).
- In the *formulation*, we specify constraints that ensure that the feasible solutions to the resulting mathematical optimization problem are indeed "feasible" in terms of the conceptual model.
- Integer (and other) variables that don't appear in the conceptual model may be introduced to enforce logical conditions.
- We also try to account for "solvability."
- We may have to prove formally that the resulting formulation does in fact correspond to the model (and eventually to the real-world problem)

Formal Definition

- Suppose $\mathcal{F} \subseteq \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p}$ is a set describing the solutions to our conceptual model.
- Then

$$\mathcal{S} = \left\{(x, y) \in (\mathbb{Z}^p \times \mathbb{R}_+^{n-p}) \times (\mathbb{Z}_+^t \times \mathbb{R}_+^{r-t}) \mid Ax + Gy \leq b\right\}$$

is a *valid (linear) formulation* if $\mathcal{F} = \mathrm{proj}_x(\mathcal{S})$.

- Note that the formulation may have auxiliary variables that are not in the conceptual model (we will see an example later in the lecture).
- A typical mathematical model can have many valid formulations.
- In this class, we will focus on problems that have linear formulations (naturally, not every problem does).
- We will see that the specific formulation we choose can have a big impact on the efficiency of the solutions method.
- Finding a "good" formulation is critical to solving a given linear model efficiently and is a good deal of what this course is about.

Proving Correctness

- There are two parts to proving a formulation is correct, although one of both of these may be “obvious”.
  - First, we have to prove that $\mathcal{F}$ is in fact the set of solutions to the original problem, which may have been described non-mathematically.
  - Second, we have to prove our formulation is correct.
- Proving correctness of a given formulation generally means proving $\mathcal{F} = \text{proj}_x(\mathcal{S})$.
- The most straightforward way of doing this involves proving
  - $x \in \mathcal{F} \Rightarrow x \in \text{proj}_x(\mathcal{S})$, and
  - $x \in \text{proj}_x(\mathcal{S}) \Rightarrow x \in \mathcal{F}$.

Problem Reduction

- Modeling involves transformation of a problem described in one formal (or informal) language into an equivalent problem described in another.
- Such transformations are formally known as *reductions* and we will study them in more detail later in the course.
- Informally, reducing problem A to problem B involves showing that there is
  - a mapping of each "instance" of problem A to an "instance" of problem B, and
  - a mapping of solutions to problem B to solutions of problem A

  such that we can solve problem A correctly by

  1. Mapping the instance of problem A to an instance of problem B;
  2. Solving the instance of problem B; and then
  3. Mapping the solution we obtain back to a solution of problem A.

Problem Reduction and Modeling

- Modeling of a general optimization problem involves reducing that model to a mathematical optimization problem.
- Proving a formulation correct amounts to proving that the general optimization problem over feasible set $\mathcal{F}$ can be reduced to a mathematical optimization problem.
- We may also do reductions from one mathematical optimization problem to another in some cases.
- These reductions may involve problems defined over completely different sets of variables.

Modeling with Integer Variables

- From a practical standpoint, why do we need integer variables?

Modeling with Integer Variables

- From a practical standpoint, why do we need integer variables?
- We have seen in the last lecture that integer variable essentially allow us to introduce *disjunctive logic*
- If the variable is associated with a physical entity that is indivisible, then the value must be integer.
  - Product mix problem.
  - Cutting stock problem.
- At its heart, integrality is a kind of disjunctive constraint.
- *0-1 (binary) variables* are often used to model more abstract kinds of disjunctions (non-numerical).
  - Modeling yes/no decisions.
  - Enforcing logical conditions.
  - Modeling fixed costs.
  - Modeling piecewise linear functions.

Modeling Binary Choice

- We use binary variables to model yes/no decisions.
- Example: Integer knapsack problem
  - We are given a set of items with associated values and weights.
  - We wish to select a subset of maximum value such that the total weight is less than a constant K.
  - We associate a 0-1 variable with each item indicating whether it is selected or not.

$$
\begin{aligned}
\max \ & \sum_{j=1}^{m} c_j x_j \\
\text{s.t.} \ & \sum_{j=1}^{m} w_j x_j \le K \\
& x \in \{0,1\}^n
\end{aligned}
$$

Modeling Dependent Decisions

- We can also use binary variables to enforce the condition that a certain action can only be taken if some other action is also taken.
- Suppose x and y are binary variables representing whether or not to take certain actions.
- The constraint $x \leq y$ says "only take action x if action y is also taken".

Example: Facility Location Problem

- We are given n potential facility locations and m customers.
- There is a fixed cost c_j of opening facility j.
- There is a cost d_{ij} associated with serving customer i from facility j.
- We have two sets of binary variables.
  - y_j is 1 if facility j is opened, 0 otherwise.
  - x_{ij} is 1 if customer i is served by facility j, 0 otherwise.
- Here is one formulation:

$$
\begin{aligned}
\min \sum_{j=1}^{n} c_j y_j + \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij} & & \\
\text{s.t.} \sum_{j=1}^{n} x_{ij} = 1 & & \forall i \\
\sum_{i=1}^{m} x_{ij} \leq m y_j & & \forall j \\
x_{ij}, y_j \in \{0, 1\} & & \forall i, j
\end{aligned}
$$

Selecting from a Set

- We can use constraints of the form $\sum_{j \in T} x_j \geq 1$ to represent that at least one item should be chosen from a set T.
- Similarly, we can also model that at most one or exactly one item should be chosen.
- Example: Set covering problem
  - A set covering problem is any problem of the form

$$\begin{aligned} \min\ & c^\top x \\ \text{s.t. } & Ax \geq 1 \\ & x_j \in \{0,1\}\ \forall j \end{aligned}$$

    where A is a 0-1 matrix.
  - Each row of A represents an item from a set S.
  - Each column A_j represents a subset S_j of the items.
  - Each variable x_j represents selecting subset S_j.
  - The constraints say that $\cup_{\{j \mid x_j = 1\}} S_j = S$.
  - In other words, each item must appear in at least one selected subset.

Modeling Disjunctive Constraints

- We are given two constraints $a^\top x \geq b$ and $c^\top x \geq d$ with nonnegative coefficients.
- Instead of insisting both constraints be satisfied, we want at least one of the two constraints to be satisfied.
- To model this, we define a binary variable y and impose

$$
\begin{aligned}
a^\top x &\geq yb, \\
c^\top x &\geq (1-y)d, \\
y &\in \{0,1\}.
\end{aligned}
$$

- More generally, we can impose that at least k out of m constraints be satisfied with

$$
\begin{aligned}
&(a^i)^\top x \geq b_i y_i, \quad i \in [1..m] \\
&\sum_{i=1}^{m} y_i \geq k, \\
&y_i \in \{0,1\}
\end{aligned}
$$

Modeling a Restricted Set of Values

- We may want variable x to only take on values in the set $\{a_1, \ldots, a_m\}$.
- We introduce m binary variables $y_j, j = 1, \ldots, m$ and the constraints

$$x = \sum_{j=1}^{m} a_j y_j,$$

$$\sum_{j=1}^{m} y_j = 1,$$

$$y_j \in \{0, 1\}$$

Piecewise Linear Cost Functions

- We can use binary variables to model arbitrary piecewise linear cost functions.
- The function is specified by ordered pairs $(a_i, f(a_i))$ and we wish to evaluate it at a point x.
- We have a binary variable y_i, which indicates whether $a_i \leq x \leq a_{i+1}$.
- To evaluate the function, we will take linear combinations $\sum_{i=1}^{k} \lambda_i f(a_i)$ of the given functions values.
- This only works if the only two nonzero $\lambda_i' s$ are the ones corresponding to the endpoints of the interval in which x lies.

Minimizing Piecewise Linear Cost Functions

- The following formulation minimizes the function.

$$
\begin{aligned}
\min \sum_{i=1}^{k} & \lambda_i f(a_i) \\
\text{s.t. } \sum_{i=1}^{k} & \lambda_i = 1, \\
& \lambda_1 \leq y_1, \\
& \lambda_i \leq y_{i-1} + y_i, \quad i \in [2..k-1], \\
& \lambda_k \leq y_{k-1}, \\
& \sum_{i=1}^{k-1} y_i = 1, \\
& \lambda_i \geq 0, \\
& y_i \in \{0, 1\}.
\end{aligned}
$$

- The key is that if $y_j = 1$, then $\lambda_i = 0, \; \forall i \neq j, j+1$.

Modeling General Nonconvex Functions

- One way of dealing with general nonconvexity is by dividing the domain of a nonconvex function into regions over which it is convex (or concave).
- We can do this using integer variables to choose the region.
- This is precisely what is done in the case of the piecewise linear cost function above.
- Most methods of general global optimization use some form of this approach.

Fixed-charge Problems

- In many instances, there is a fixed cost and a variable cost associated with a particular decision.
- Example: Fixed-charge Network Flow Problem
  - We are given a directed graph $G = (N, A)$.
  - There is a fixed cost c_{ij} associated with "opening" arc (i, j) (think of this as the cost to "build" the link).
  - There is also a variable cost d_{ij} associated with each unit of flow along arc (i, j).
  - Consider an instance with a single supply node.
    * Minimizing the fixed cost by itself is a minimum spanning tree problem (easy).
    * Minimizing the variable cost by itself is a minimum cost network flow problem (easy).
    * We want to minimize the sum of these two costs (difficult).

Modeling the Fixed-charge Network Flow Problem

- To model the FCNFP, we associate two variables with each arc.
  - x_{ij} (*fixed-charge variable*) indicates whether arc (i, j) is open.
  - f_{ij} (*flow variable*) represents the flow on arc (i, j).
  - Note that we have to ensure that $f_{ij} > 0 \Rightarrow x_{ij} = 1$.

$$
\begin{aligned}
\min \sum_{(i,j)\in A} & c_{ij}x_{ij} + d_{ij}f_{ij} & \\
\text{s.t.} \sum_{j\in O(i)} f_{ij} - \sum_{j\in I(i)} & f_{ji} = b_i & \forall i \in N \\
& f_{ij} \leq C x_{ij} & \forall (i,j) \in A \\
& f_{ij} \geq 0 & \forall (i,j) \in A \\
& x_{ij} \in \{0,1\} & \forall (i,j) \in A
\end{aligned}
$$

Alternative Formulations

- Recall our earlier definition of a valid formulation.
- A key concept in the rest of the course will be that every mathematical model has many alternative formulations.
- Many of the key methodologies in integer programming are essentially automatic methods of reformulating a given model.
- The goal of the reformulation is to make the model easier to solve.

Simple Example: Knapsack Problem

- We are given a set $N = \{1, \ldots n\}$ of items and a capacity W.
- There is a profit p_i and a size w_i associated with each item $i \in N$.
- We want to choose the set of items that maximizes profit subject to the constraint that their total size does not exceed the capacity.
- The most straightforward formulation is to introduce a binary variable x_i associated with each item.
- x_i takes value 1 if item i is chosen and 0 otherwise.
- Then the formulation is

$$\begin{aligned} \min \sum_{j=1}^{n} p_j x_j & \\ \text{s.t. } \sum_{j=1}^{n} w_j x_j \leq W & \\ x_i \in \{0, 1\} & \qquad \forall i \end{aligned}$$

- Is this formulation correct?

An Alternative Formulation

- Let us call a set $C \subseteq N$ a *cover* is $\sum_{i \in C} w_i > W$.
- Further, a cover C is *minimal* if $\sum_{i \in C \setminus \{j\}} w_i > W$ for all $j \in C$.
- Then we claim that the following is also a valid formulation of the original problem.

$$\begin{aligned} \min \sum_{j=1}^{n} p_j x_j & \\ \text{s.t.} \sum_{j \in C} x_j \leq |C| - 1 & \quad \text{for all minimal covers } C \\ x_i \in \{0, 1\} & \quad i \in N \end{aligned}$$

- Which formulation is "better"?

Back to the Facility Location Problem

- Recall our earlier formulation of this problem.
- Here is another formulation for the same problem:

$$
\begin{aligned}
\min \; & \sum_{j=1}^{n} c_j y_j + \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij} & \\
\text{s.t.} \; & \sum_{j=1}^{n} x_{ij} = 1 & \forall i \\
& x_{ij} \leq y_j & \forall i, j \\
& x_{ij}, y_j \in \{0, 1\} & \forall i, j
\end{aligned}
$$

- Notice that the set of integer solutions contained in each of the polyhedra is the same (why?).
- However, the second polyhedron is strictly included in the first one (how do we prove this?).
- Therefore, the second polyhedron will yield a better lower bound.
- The second polyhedron is a better approximation to the convex hull of integer solutions.

Formulation Strength and Ideal Formulations

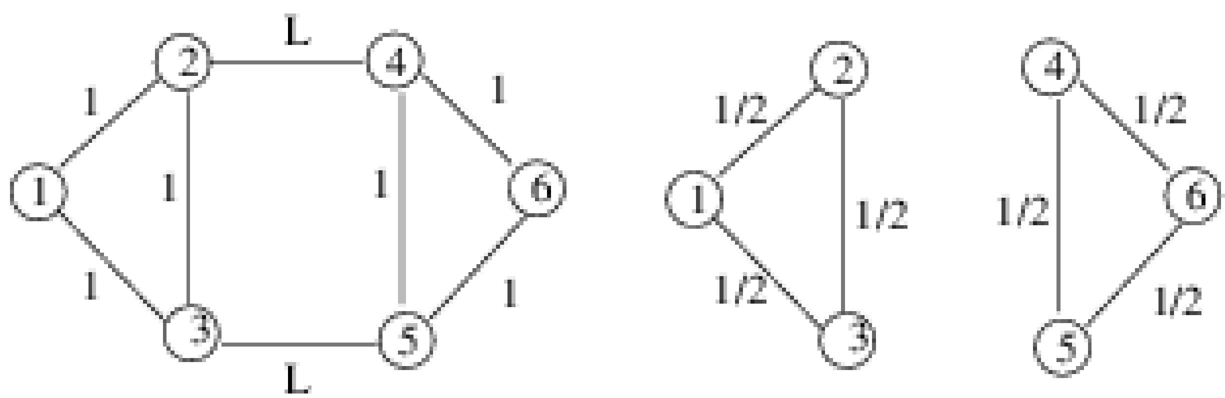
- Consider two formulations A and B for the same ILP.
- Denote the feasible regions corresponding to their LP relaxations as $\mathcal{P}_A$ and $\mathcal{P}_B$.
- Formulation A is said to be *at least as strong as* formulation B if $\mathcal{P}_A \subseteq \mathcal{P}_B$.
- If the inclusion is strict, then A is *stronger than* B.
- If $\mathcal{S}$ is the set of all feasible integer solutions for the ILP, then we must have $conv(\mathcal{S}) \subseteq \mathcal{P}_A$ (why?).
- A is *ideal* if $conv(\mathcal{S}) = \mathcal{P}_A$.
- If we know an ideal formulation, we can solve the IP (why?).
- How do our formulations of the knapsack problem compare by this measure?

Strengthening Formulations

- Often, a given formulation can be strengthened with additional inequalities satisfied by all feasible integer solutions.
- Example: The Perfect Matching Problem
  - We are given a set of n people that need to paired in teams of two.
  - Let c_{ij} represent the "cost" of the team formed by person i and person j.
  - We wish to maximize efficiency over all teams.
  - We can represent this problem on an undirected graph $G = (N, E)$.
  - The nodes represent the people and the edges represent pairings.
  - We have $x_e = 1$ if the endpoints of e are matched, $x_e = 0$ otherwise.

$$
\begin{aligned}
\min \quad & \sum_{e=\{i,j\}\in E} c_e x_e \\
\text{s.t.} \quad & \sum_{\{j \mid \{i,j\}\in E\}} x_{ij} = 1, \ \forall i \in N \\
& x_e \in \{0,1\}, \qquad \forall e = \{i,j\} \in E.
\end{aligned}
$$

Valid Inequalities for Matching



- Consider the graph on the left above.
- The optimal perfect matching has value $L + 2$.
- The optimal solution to the LP relaxation has value 3.
- This formulation can be extremely weak.
- Add the *valid inequality* $x_{24} + x_{35} \geq 1$.
- Every perfect matching satisfies this inequality.

The Odd Set Inequalities

- We can generalize the inequality from the last slide.
- Consider the cut S corresponding to any odd set of nodes.
- The *cutset* corresponding to S is

$$\delta(S) = \{\{i, j\} \in E | i \in S, j \notin S\}.$$

- An *odd cutset* is any $\delta(S)$ for which $|S|$ is odd.
- Note that every perfect matching contains at least one edge from every odd cutset.
- Hence, each odd cutset induces a possible valid inequality.

$$\sum_{e \in \delta(S)} x_e \geq 1, S \subset N, |S| \text{ odd.}$$

Using the New Formulation

- If we add all of the odd set inequalities, the new formulation is ideal.
- Hence, we can solve this LP and get a solution to the IP.
- However, the number of inequalities is exponential in size, so this is not really practical.
- Recall that only a small number of these inequalities will be active at the optimal solution.
- Later, we will see how we can efficiently generate these inequalities on the fly to solve the IP.

Extended Formulations

- We have so far focused on strengthening formulations using additional constraints.
- However, changing the set of variables can also have a dramatic effect.
- Example: A Lot-sizing Problem
  - We want to minimize the costs of production, storage, and set-up.
  - Data for period $t = 1, \ldots, T$:
    * d_t: total demand,
    * c_t: production set-up cost,
    * p_t: unit production cost,
    * h_t: unit storage cost.
  - Variables for period $t = 1, \ldots, T$:
    *
    *
    *

Lot-sizing: The "natural" formulation

- Here is the formulation based on the "natural" set of variables:

$$\begin{aligned}
\min \sum_{t=1}^{T} & (p_t y_t + h_t s_t + c_t x_t) \\
\text{s.t. } & y_1 = d_1 + s_1, \\
& s_{t-1} + y_t = d_t + s_t, && \text{for } t = 2, \ldots, T, \\
& y_t \leq \omega_t x_t, && \text{for } t = 1, \ldots, T, \\
& s_T = 0, \\
& s, y \in \mathbb{R}^T_+, \\
& x \in \{0,1\}^T.
\end{aligned}$$

- Here, $\omega_t = \sum_{i=t}^{T} d_i$ is an upper bound on y_t.

Lot-sizing: The “extended” formulation

- Suppose we split the production lot in period t into smaller pieces.
- Define the variables q_{ti} to be the production in period t designated to satisfy demand in period $i \geq t$.
- Now, $y_t = \sum_{i=t}^{T} q_{ti}$.
- With the new set of variables, we can impose the tighter constraint

$$q_{ti} \leq d_i x_t \text{ for } i = 1, \ldots, T \text{ and } t = 1, \ldots, T.$$

- The additional variables strengthen the formulation.
- Again, this in contrary to conventional wisdom for formulating linear programs.

Strength of Formulation for Lot-sizing

- Although the formulation from the previous slide is much stronger than our original, it is still not ideal.
- Consider the following sample data.

```
# The demands for six periods
DEMAND = [6, 7, 4, 6, 3, 8]

# The production cost for six periods
PRODUCTION_COST = [3, 4, 3, 4, 4, 5]

# The storage cost for six periods
STORAGE_COST = [1, 1, 1, 1, 1, 1]

# The set up cost for six periods
SETUP_COST = [12, 15, 30, 23, 19, 45]

# Set of periods
PERIODS = range(len(DEMAND))
```

Strength of Formulation for Lot-sizing (cont'd)

```
Optimal Total Cost is:  171.42016761

Period  0 :  13 units produced,  7 units stored, 6 units sold
0.38235294 is the value of the fixed charge variable
Period  1 :  0 units produced,  0 units stored, 7 units sold
0.0 is the value of the fixed charge variable
Period  2 :  4 units produced,  0 units stored, 4 units sold
0.19047619 is the value of the fixed charge variable
Period  3 :  6 units produced,  0 units stored, 6 units sold
0.35294118 is the value of the fixed charge variable
Period  4 :  11 units produced, 8 units stored, 3 units sold
1.0 is the value of the fixed charge variable
Period  5 :  0 units produced,  0 units stored, 8 units sold
0.0 is the value of the fixed charge variable
```

What is happening here?

Strength of Formulation for Lot-sizing (cont'd)

Let's take a more detailed look:

```
production in period 0 for period 0 : 2.2941176
production in period 0 for period 1 : 2.6764706
production in period 0 for period 2 : 1.5294118
production in period 0 for period 3 : 2.2941176
production in period 0 for period 4 : 1.1470588
production in period 0 for period 5 : 3.0588235
```

What is the problem?

An Ideal Formulation for Lot-sizing

- We can further strengthen the formulation by adding the constraint

$$\sum_{t=1}^{i} q_{ti} \geq d_i \text{ for } i = 1, \ldots, T$$

- In fact, adding these additional constraints makes the formulation ideal.
- If we *project* into the original space, we will get the convex hull of solutions to the first formulation.
- How would we prove this?

Contrast with Linear Programming

- In linear programming, the same problem can also have multiple formulations.
- In LP, however, conventional wisdom is that bigger formulations take longer to solve.
- In IP, this conventional wisdom does not hold.
- We have already seen two examples where it is not valid.
- Generally speaking, the size of the formulation does not determine how difficult the IP is.