Valid inequalities for MIPs and group polyhedra from approximate liftings

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Overview

1. Lifting and the Group Problem
2. A Lifting Procedure to Generate Cuts
   - Relation to the Group Problem
   - CPL_n Functions
   - New Facets of the Group Problem
3. Conclusion and Future Work

The Mixed Integer Knapsack Polyhedron

Consider:

$$PS = conv\{(x,y) \in \mathbb{Z}_+^m \times [0,1]^n | \sum_{i\in M} a_i x_i + \sum_{j \in N} b_j y_j \leq a_0$$

with integer data and $a_1 \neq 0$.

Let $PS = conv(S)$.

Generating group cuts for an integer PS

1. Choose an integer K.
2. Obtain remainder r_j: $a_j = Kq_j + r_j$.
3. Relax PS:

$$G = \{x \in \mathbb{Z}_+^m | \sum_{i \in M} r_i x_i \equiv r_0 \pmod{K}\},$$

4. Relax G into the master cyclic group polyhedron

$$P(C_{K,r_0}) = Conv\{x \in \mathbb{Z}_+^{K-1} | \sum_{i=1}^{K-1} i x_i \equiv r_0 \text{ (mod } K)\}.$$

5. Use the facets of $P(C_{K,r_0})$ as valid inequalities for PS.

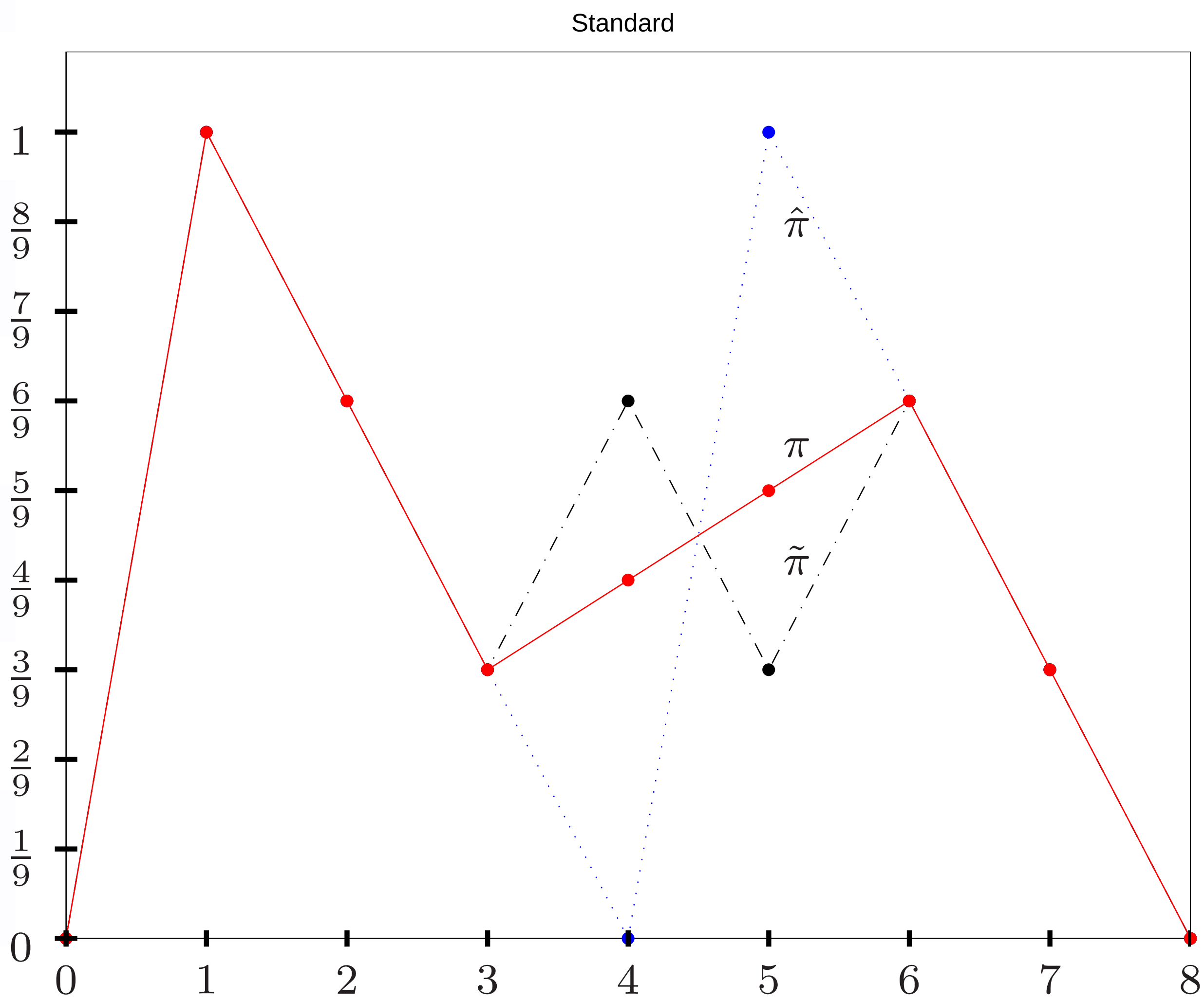
Subadditive Characterization of Facets

Theorem [Gomory 69]: For $1 \leq r_0 \leq K-1$, the non-trivial facet-defining inequalities $\sum_{i=1}^{K-1} \pi_i x_i \geq \gamma$ of the master cyclic group polyhedron $P(C_{K,r_0})$ are given by the extreme rays of the cone S_{K,r_0} defined by $\pi \in \mathbb{R}^n$ such that:

Nonnegativity:	$\pi_i \geq 0,$	$1 \leq i \leq K-1,$
Subadditivity:	$\pi_i + \pi_j \geq \pi_k,$	$1 \leq i,j,k \leq K-1,$
		$(i+j) \equiv k \text{ (mod } K),$
Complementarity:	$\pi_i + \pi_j = \gamma,$	$1 \leq i,j \leq K-1,$
		$(i+j) \equiv r_0 \text{ (mod } K),$
Scalability:	$\pi_{r_0} = \gamma.$	

Representing Facets of the Group Problem

3 Facets of $P(C_{8,1})$



Issues with the Group Problem

Discrete Group: (K is lcd of a_i's)

1. Can generate a cut by solving a Linear Program. (+)
2. Does not require the explicit derivation of cuts (+)
3. In practice, K is large and difficult to obtain. (-)

Continuous Group: ($K = 1$)

1. Does not require the determination of K. (+)
2. The "LP" to solve has an infinite number of variables and constraints (-)
3. Requires the explicit derivation of cuts (-)

Generating Cuts through Lifting: Notation

Let
M_0, be a subset of M
N_0, N_1 be non intersecting subsets of N.

Define $PS(M_0, N_0, N_1)$ as :

$$conv\{(x,y) \in \mathbb{R}^{m+n} \mid \sum_{j \in M} a_j x_j + \sum_{j \in N} b_j y_j = a_0$$
$$x_j = 0 \, \forall j \in M_0, y_j = 0 \, \forall j \in N_0$$
$$y_j = 1 \, \forall j \in N_1\}$$

Ex : $PS_I = PS(\emptyset, N, \emptyset)$ *is an integer polytope.*

Step 1: Initial Inequality

For $K > 0$ define the polyhedron

$$PS' = conv\{(x,y) \in \mathbb{Z}^m \times [0,1]^n \mid \sum_{j \in M}(Kq_j + r_j)x_j + \sum_{j \in N^+} b_j y_j + \sum_{j \in N^-}(-b_j)\bar{y}_j = Kq_0 + r_0 + \sum_{j \in N^-}(-b_j)\}$$

where $r_j < K, \forall j \in M$.

(PS' is equivalent to PS.)

Step 1: Initial Inequality

The defining inequality of $PS'(M\backslash\{1\}, N^+, N^-)$ is

$$(Kq_1 + r_1)x_1 = Kq_0 + r_0$$

- By dividing this inequality by K and rounding, we see that

$$q_1 x_1 \leq q_0 \qquad (1)$$

  is valid for $PS'(M\backslash\{1\}, N^+, N^-)$.
- Inequality (1) is not necessarily valid for PS'
- It must be *lifted* into a valid inequality of PS'

Step 2: Integer Lifting

Theorem [Wolsey]: For $i = 1, \ldots, m$, let

$$\Phi^i(a) = q_0 - \max \{q_1 x_1 + \sum_{j=2}^{i-1} \Phi^{j-1}(Kq_j + r_j)x_j\}$$

$$s.t. \quad (Kq_1 + r_1)x_1 + \sum_{j=2}^{i-1}(Kq_j + r_j)x_j = Kq_0 + r_0 - a$$

Then the inequality

$$q_1 x_1 + \sum_{j \in M \setminus \{1\}} \Phi^{j-1}(a_j)x_j \leq q_0$$

is valid for $PS(\emptyset, N_0, N_1)$.

Step 2: Integer Lifting

1. There is not an easy closed form expression for Φ^i.
2. The function Φ^i can be computed in pseudo-polynomial time.
3. The lifting function needs to be recomputed after any variable is lifted.
4. To obtain the lifting coefficients quickly, we use approximate integer lifting (Wolsey, Gu et al., Atamturk).

Step 2: Integer Lifting

For $q_1 > 0$,

$$\Phi^1(a) = \begin{cases} q_0 - q_1 \left\lfloor \frac{Kq_0 + r_0 - a}{Kq_1 + r_1} \right\rfloor & \text{if } \quad a \leq Kq_0 + r_0 \\ \infty & \text{if } \quad a > Kq_0 + r_0. \end{cases}$$

For $q_1 < 0$,

$$\Phi^1(a) = \begin{cases} q_0 - q_1 \left\lceil \frac{Kq_0 + r_0 - a}{Kq_1 + r_1} \right\rceil & \text{if } \quad a \geq Kq_0 + r_0 \\ q_0 & \text{if } \quad a < Kq_0 + r_0. \end{cases}$$

Step 2: First Lower Approximation

- Find a continuous function that approximates Φ^1 from below and depends only on r_0:

$$\Phi(a) := \lceil \frac{a - r_0}{K} \rceil$$

- Next, find a *superadditive* function that approximates Φ from below.

Step 2: Superadditive Approximation

A function $\phi : \mathbb{R} \to \mathbb{R}$ is superadditive if $\phi(a) + \phi(b) \leq \phi(a+b)$ for $a, b \in \mathbb{R}$.

Theorem [Wolsey]: Assume that $\phi(a) \leq \Phi^1(a)$ for $a \in \mathbb{R}$, then

$$q_1 x_1 + \sum_{j \in M \setminus \{1\}} \phi(a_j) x_j \leq q_0$$

is valid for $PS(\emptyset, N_0, N_1)$.

Step 2: Strong Approximation Functions

1. Validity: $\phi(a) \leq \Phi(a), \forall a \in \mathbb{R}$
2. Superadditivity:

$$\phi(a) + \phi(b) \leq \phi(a+b),\ \forall a, b \in \mathbb{R}$$

3. Pseudo-Periodicity:

$$\phi(a+K) = 1 + \phi(a),\ \forall a \in \mathbb{R}$$

4. Pseudo-Symmetry:

$$\begin{aligned} \phi(a) &= 0, \forall a \in [0, r_0], \\ \phi(r_0 + \epsilon) &= 1 - \phi(K - \epsilon), \forall \epsilon \in [0, K - r_0] \end{aligned}$$

Step 2: Integer-Lifted Rounding Cut

If ϕ satisfies the validity, superadditivity, pseudo-periodicity, and pseudo-symmetry properties then

- ϕ is not dominated by any other valid superadditive function
- and

$$q_1 x_1 + \sum_{j \in M \setminus \{1\}} (q_j + \phi(r_j))x_j \leq q_0 \qquad (2)$$

is valid for $PS(\emptyset, N^+, N^-)$

Step 3: Continuous Lifting

Assume that $\phi'_+(r_0) = \lim_{\epsilon \to 0^+} \frac{\phi(r_0+\epsilon)}{\epsilon}$ exists. Then

$$q_1 x_1 + \sum_{j \in M \setminus \{1\}} (q_j + \phi(r_j)) x_j + \sum_{j \in N^-} \phi'_+(r_0) b_j \bar{y}_j \leq q_0$$

is a valid inequality for PS.

Relations to the Group Problem

- Assume that $a_j \in \mathbb{Z}, \forall j \in M$, $K \in \mathbb{N}$.
- Any inequality valid for PS is valid for

$$PQ = conv\{(x,y) \in \mathbb{R}^{m+n} \quad | \quad \sum_{j \in M} (Kq_j + r_j)x_j = Kq_0 + r_0$$
$$x_j \in \mathbb{N} \qquad \forall j \in M\}.$$

- $\sum_{j \in M} \frac{r_j - K\phi(r_j)}{r_0} x_j \geq 1$ is valid for PQ.

Relations to the Group Problem

The function $f(u) = \frac{r(u) - K\phi(r(u))}{r_0}$ satisfies

1. $f(u) \geq 0$, $\forall u \in [0, K]$,
2. $f(u) = \frac{u}{r_0}$, $\forall u \in [0, r_0]$,
3. $f(u) + f(v) \geq f((u + v) \mod K)$, and
4. $f(u) + f((r_0 - u) \mod K) = f(r_0)$ for $u \in [0, K]$.

Relations to the Group Problem

Therefore

$$\sum_{j=1}^{K-1} f(j/K)t_j \geq f(r_0/K)$$

is a valid inequality for the master cyclic group polyhedron

$$P(C_{K,r_0}) = conv\{t \in Z^n_+ | \sum_{j=1}^{K-1} jt_j \equiv r_0 \mod K\}.$$

Deriving Facets of the Group Problem

- How can we use this procedure to derive strong inequalities for the group problem?
- Consider a "nice" family of parameterized lifting functions.
- For which parameters these functions are the strongest (in the lifting space)?
- Are the resulting inequalities are also strong for the group problem?

CPL_n Functions

For $K \in \mathbb{R}_+$, $r_0 \in (0, K)$, $n \in \mathbb{Z}_+$, $z = (z_1, ..., z_n) \in \mathbb{R}^n_+$, and $\theta = (\theta_1, ..., \theta_n) \in \mathbb{R}^n_+$ such that $\sum_{i=1}^n z_i = \frac{K - r_0}{2}$ and $\sum_{i=1}^n \theta_i = 1/2$, a pseudo-periodic function $\phi(a)$ is a CPL$_n$ function if, when a is restricted in $[0, K)$,

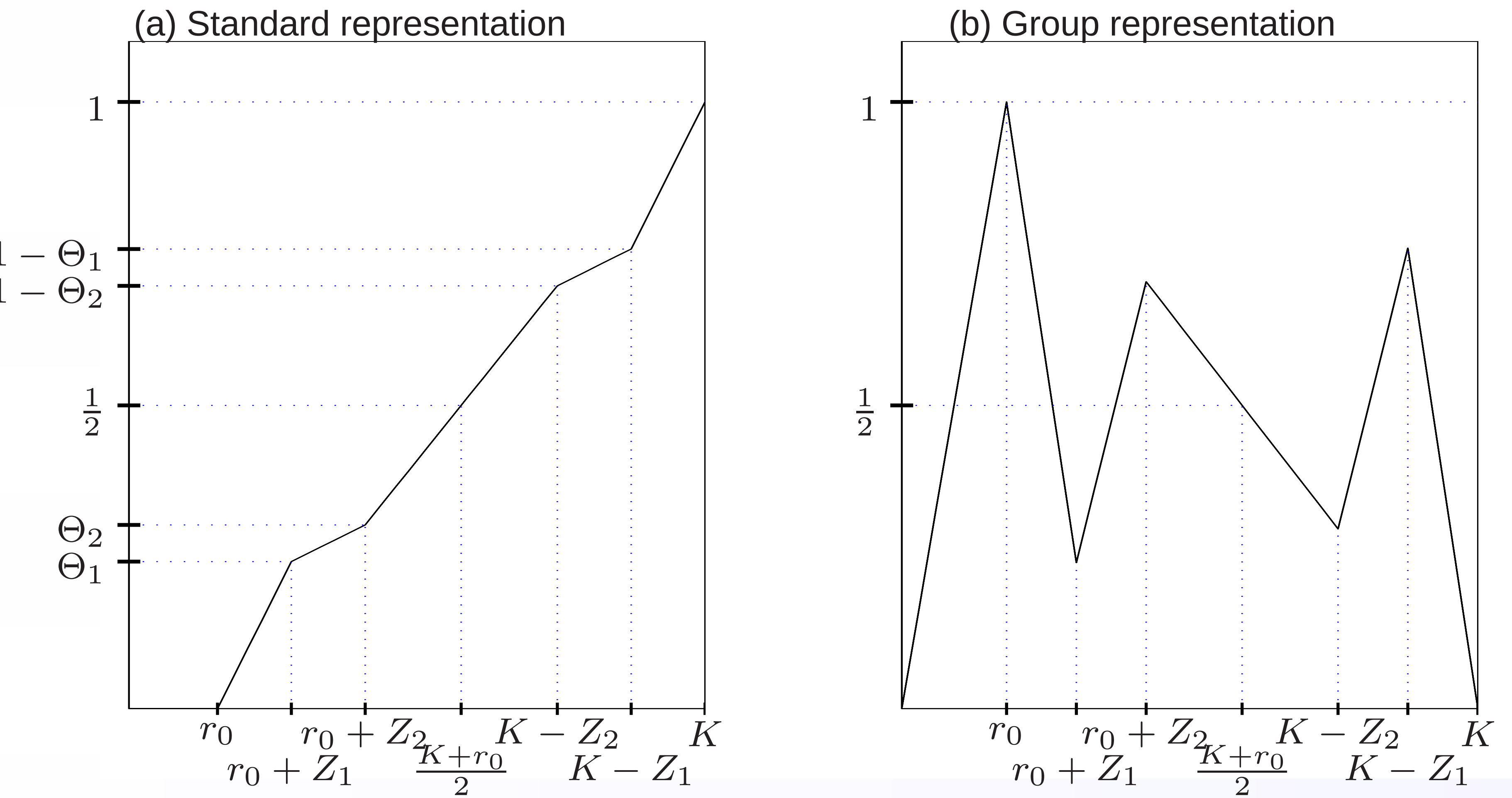
$$\phi(a) = \begin{cases} 0, & \text{if} \quad a \in [0, r_0], \\ \Theta_{i-1} + \frac{\theta_i}{z_i}(v - r_0 - Z_{i-1}), & \text{if} \quad a \in (r_0 + Z_{i-1}, r_0 + Z_i], \\ 1 - \Theta_i + \frac{\theta_i}{z_i}(v - K + Z_i), & \text{if} \quad a \in (K - Z_i, K - Z_{i-1}], \end{cases}$$

where $Z_0 = 0$, $Z_i = \sum_{j=1}^i z_i$, $\theta_0 = 0$, and $\Theta_i = \sum_{j=1}^i \theta_i$. □

CPL_n Functions

1. CPL$_n$ functions are valid, pseudo-periodic, pseudo-symmetric.
2. CPL$_n$ functions are continuous.
3. CPL$_n$ functions are piecewise-linear over $2n$ intervals.

Example: A CPL_3 Function



CPL_n Inequalities: Superadditivity Conditions

A CPL$_n$ function $\phi(a)$ is superadditive if and only if

$$\begin{aligned}
\phi(r_0 + Z_i) + \phi(r_0 + Z_j) &\leq \phi(2r_0 + Z_i + Z_j), && 0 \leq i \leq j \leq n-1, \\
\phi(r_0 + Z_i) + 1 &\leq \phi(r_0 + K + Z_i - Z_j) + \phi(r_0 + Z_j), && 0 \leq i, j \leq n-1, \\
\phi(r_0 + Z_i + Z_j) &\leq \phi(r_0 + Z_i) + \phi(r_0 + Z_j), && 0 \leq i \leq j \leq n-1.
\end{aligned}$$

- Only a finite number of points must be checked.
- All relations are linear in θ for fixed z.

Superadditive CPL_n functions

For valid z, θ defines a superadditive CPL$_n$ function if and only if θ belongs to the polyhedron

$$
\begin{aligned}
P\Theta_n(z) := \{\theta \in \mathbb{R}^{n-1}_+ \,|\, & \Theta_i + \Theta_j \le \phi(2r_0 + Z_i + Z_j), && 0 \le i, j \le n-1, \\
& \Theta_i - \Theta_j \le \phi(r_0 + K + Z_i - Z_j) - 1, && 0 \le i, j \le n-1, \\
& \Theta_i + \Theta_j \ge \phi(r_0 + Z_i + Z_j), && 0 \le i, j \le n-1, \\
& \Theta_{n-1} \le \frac{1}{2} \,\}. && \square
\end{aligned}
$$

- All "extreme" superadditive CPL_n functions correspond to extreme points of $P\Theta_n(z)$.

Example: CPL_2 Functions

$$P\Theta_2(z_1) = \{\theta_1 \in \mathbb{R}_+ \,|\, \phi(r_0 + 2z_1) \leq 2\theta_1 \leq \phi(2r_0 + 2z_1)\}.$$

The following are the only extreme points of $P\Theta_2(z_1)$:

1. $z_1 \in [0, \frac{K-r_0}{2}] \Rightarrow \theta_1^1 = \frac{z_1}{K-r_0}$ (GMIC)
2. $z_1 \in [0, \frac{K-2r_0}{3}) \Rightarrow \theta_1^2 = \frac{z_1+r_0}{K+r_0}$ (2-Slope)
3. $z_1 \in [\frac{K-2r_0}{3}, \frac{K-2r_0}{2}) \Rightarrow \theta_1^3 = \frac{z_1}{K-2r_0}$ (3-Slope)
4. $z_1 \in [\frac{K-2r_0}{2}, \frac{K-r_0}{2}] \Rightarrow \theta_1^4 = \frac{1}{2}$ (new 3-Slope)

CPL_3-Extreme Functions

1. $n = 2$ was interesting. What about $n = 3$?
2. Too many cases to analyze by hand.
3. Restrict to $z_1 = z_2$. Only 53 cases!

$CPL_3^=$ Functions

For $r_0 + 4z_1 \leq K$,

$$
\begin{aligned}
P\Theta_3^=(z_1) = \{(\theta_1, \theta_2) \in \mathbb{R}^2 \quad | \quad & \theta_2 \geq -\phi(r_0 - z_1) \\
& 2\theta_1 \leq \phi(2r_0 + 2z_1) \\
& 2\theta_1 + \theta_2 \geq \phi(r_0 + 3z_1) \\
& 2\theta_1 + \theta_2 \leq \phi(2r_0 + 3z_1) \\
& 2\theta_1 + 2\theta_2 \geq \phi(r_0 + 4z_1) \\
& 2\theta_1 + 2\theta_2 \leq \phi(2r_0 + 4z_1) \\
& \theta_1 - \theta_2 \geq 0 \\
& \theta_1 \geq 0, \theta_2 \geq 0\}.
\end{aligned}
$$

- Only 18 unique extreme points!

$CPL_3^=$-Extreme Functions: A Summary

Extreme point	θ_1	θ_2	Range of r_0	Range of K
a	$\frac{z_1}{K-r_0}$	$\frac{z_1}{K-r_0}$	all	all
b	$\frac{r_0+2z_1}{2K+2r_0}$	$\frac{r_0+2z_1}{2K+2r_0}$	all	$2r_0+6z_1 \leq K$
c	$\frac{r_0+z_1}{K+r_0}$	$\frac{z_1}{K+r_0}$	all	$2r_0+4z_1 < K$
d	$\frac{r_0+2z_1}{2K-2r_0}$	$\frac{2z_1-r_0}{2K-2r_0}$	$0 < r_0 \leq 2z_1$	$r_0+6z_1 \leq K$
e	$\frac{z_1}{K-2r_0}$	$\frac{z_1}{K-2r_0}$	all	$2r_0+4z_1 \leq K < 2r_0+6z_1$
f	$\frac{-Kz_1-Kr_0+6z_1r_0+r_0^2+4z_1^2}{4Kz_1+8z_1r_0-K^2+r_0^2}$	$\frac{z_1(2r_0+4z_1-K)}{4Kz_1+8z_1r_0-K^2+r_0^2}$	$0 < r_0 < 2z_1$	$\max\{r_0+5z_1, 2r_0+4z_1\} \leq K < r_0+6z_1$
g	$\frac{z_1}{K-3r_0}$	$\frac{z_1-r_0}{K-3r_0}$	$0 < r_0 < z_1$	$2r_0+4z_1 \leq K < r_0+5z_1$
h	$\frac{1}{4}$	$\frac{1}{4}$	all	$r_0+4z_1 \leq K < 2r_0+4z_1$
i	$\frac{K-2r_0-5z_1}{2(K-2r_0-6z_1)}$	$\frac{-z_1}{2(K-2r_0-6z_1)}$	all	$\max\{r_0+4z_1, 2r_0+3z_1\} \leq K < 2r_0+4z_1$

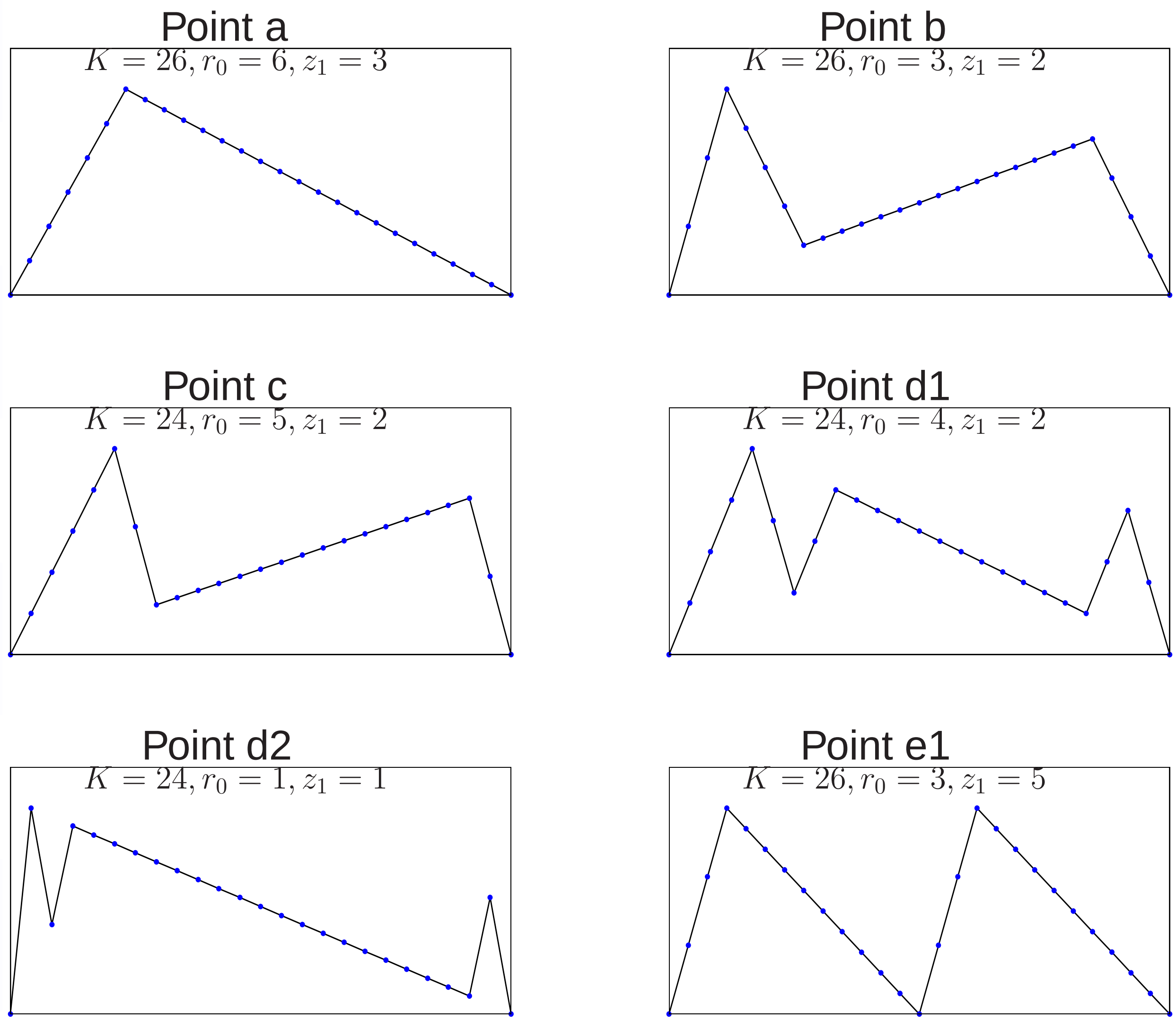
$CPL_3^=$-Extreme Functions: A Summary

Extreme point	θ_1	θ_2	Range of r_0	Range of K
j	$\frac{z_1(2r_0+5z_1-K)}{-K^2+3Kr_0+5Kz_1-2r_0^2-7z_1r_0}$	$\frac{z_1(r_0+5z_1-K)}{-K^2+3Kr_0+5Kz_1-2r_0^2-7z_1r_0}$	$0 < r_0 < 2z_1$	$\max\{r_0+4z_1, 2r_0+3z_1\} \leq K < \min\{r_0+5z_1, 2r_0+4z_1\}$
k	$\frac{1}{3}$	0	$r_0 > 0$ $r_0 > 3z_1$	$\max\{r_0+5z_1, 2r_0+3z_1\} \leq K \leq 2r_0+4z_1$ $r_0+5z_1 \leq K < r_0+6z_1$
l	$\frac{z_1}{K-2r_0}$	$\frac{2z_1-r_0}{2K-4r_0}$	$0 < r_0 \leq 2z_1$	$r_0+4z_1 \leq K < 2r_0+3z_1$
m	$\frac{r_0}{K+r_0-4z_1}$	0	$r_0 > 2z_1$	$2r_0+4z_1 < K$
n	$\frac{2z_1}{K-r_0}$	0	$r_0 > 2z_1$	$r_0+6z_1 \leq K$
o	$\frac{z_1}{K-2r_0}$	$\frac{K-2r_0-2z_1}{2K-4r_0}$	$r_0 > z_1$	$\max\{r_0+4z_1, 2r_0+2z_1\} \leq K < 2r_0+3z_1$
p	$\frac{z_1}{K-2r_0}$	0	$r_0 > 2z_1$	$2r_0+2z_1 \leq K < 2r_0+3z_1$
q	$\frac{z_1}{K-r_0-2z_1}$	0	$r_0 > 2z_1$	$r_0+4z_1 \leq K < r_0+5z_1$
r	$\frac{1}{2}$	0	$r_0 > 2z_1$	$r_0+4z_1 \leq K \leq 2r_0+2z_1$

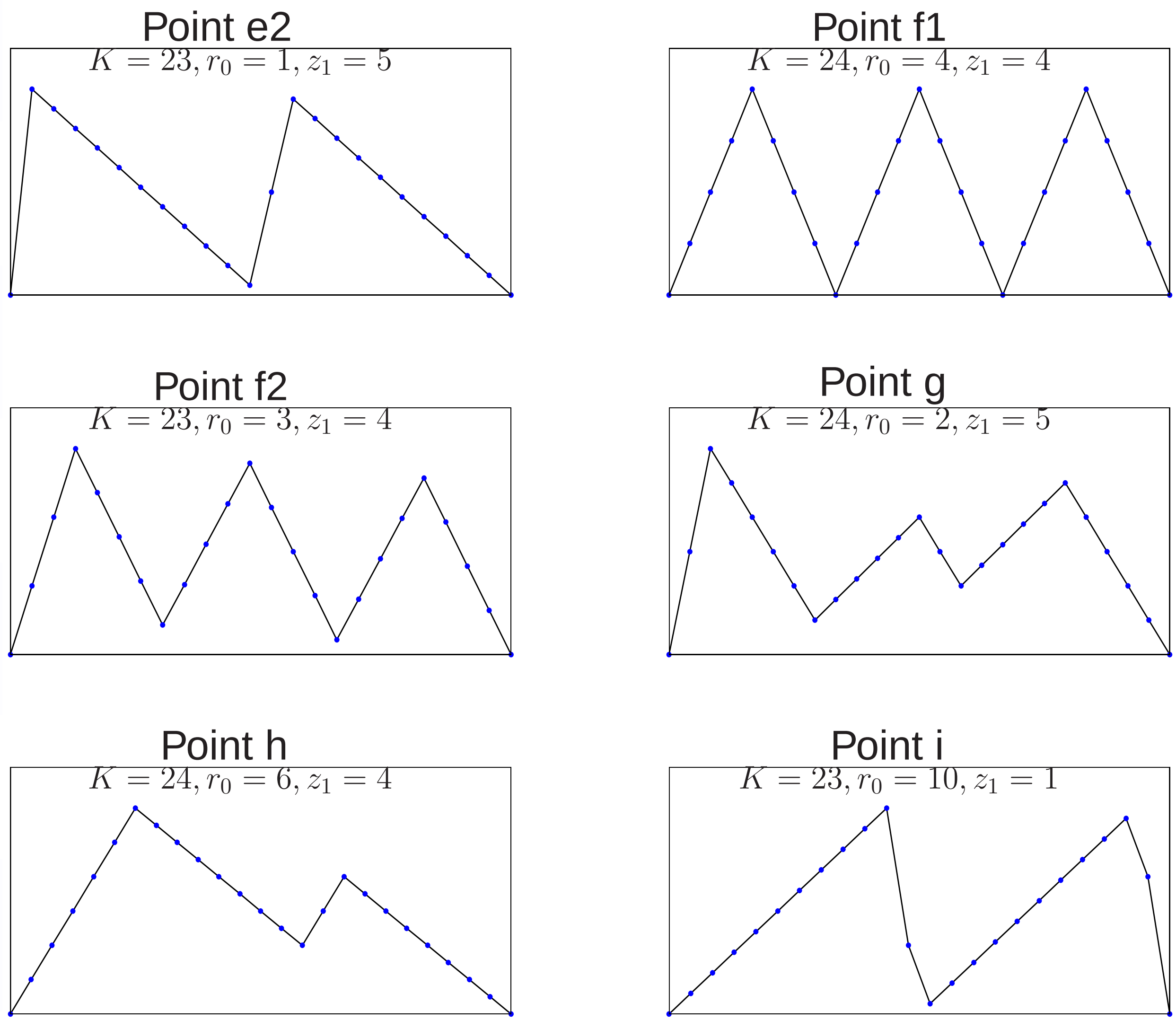
$\text{CPL}_3^=$-Extreme Functions

- These points are $\text{CPL}_3^=$-Extreme.
- Are they strong for the group problem?
- What dimension face is induced on the group polyhedron?
- When do they yield facets?

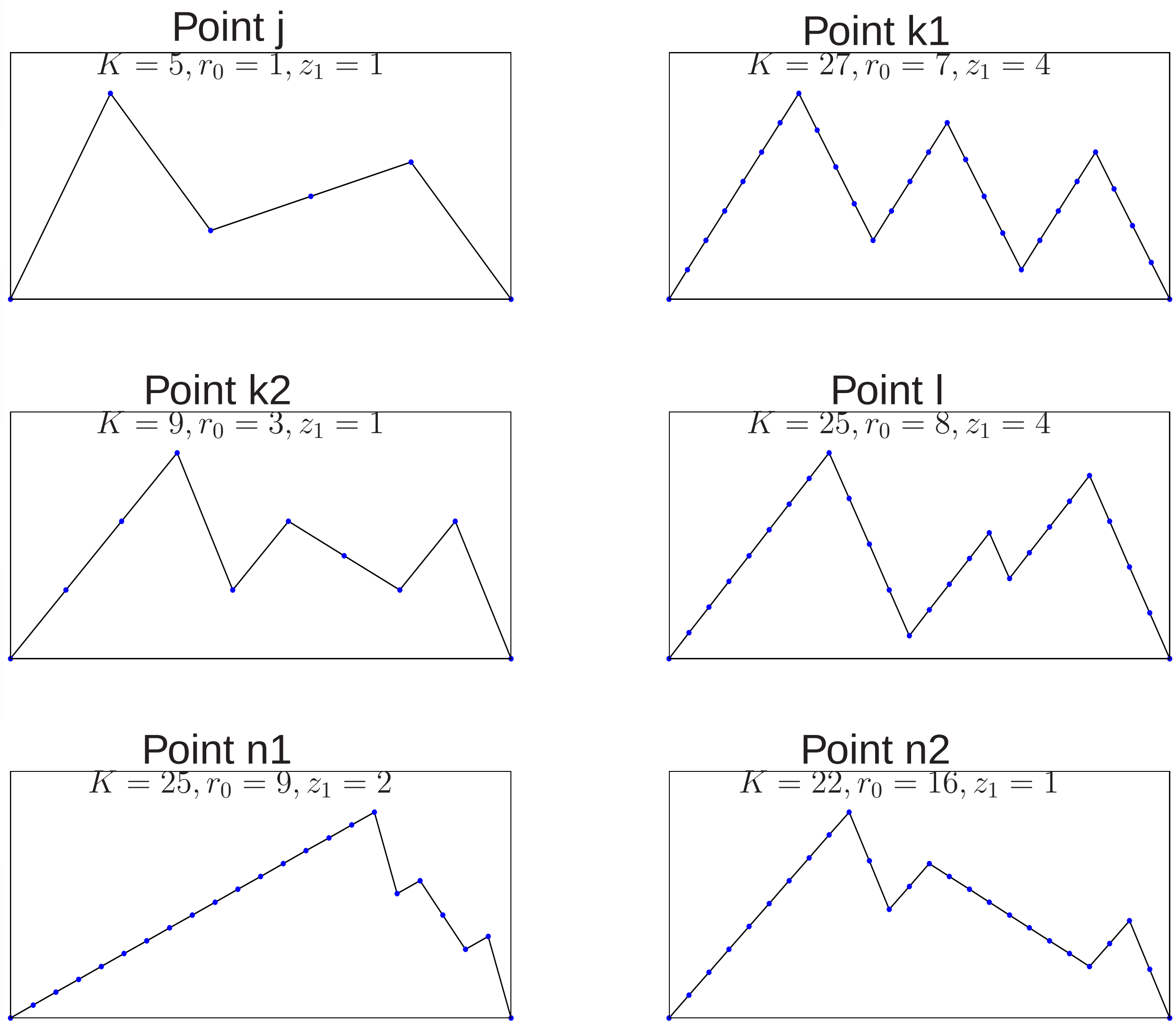
Facets of the Group Problem (1/4)



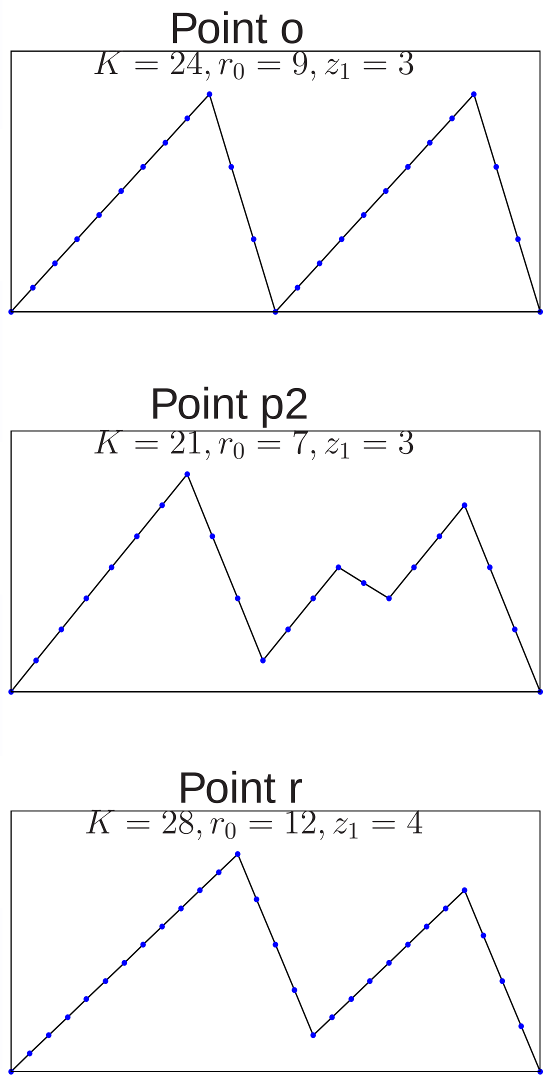
Facets of the Group Problem (2/4)



Facets of the Group Problem (3/4)



Facets of the Group Problem (4/4)



New families

- (d2): 4-slope facets
- (e2),(f2),(n2),(p2): new 3-slope facets
- (l),(q): new constructive 2-slope facets
- Many are extreme for infinite group problem

Conclusion

1. Alternative derivation of group polyhedron facets.
2. Scheme is simple and constructive.
3. Simple derivation of the GMIC, and Gomory and Johnson's 2-Slopes and 3-Slopes.
4. Derivation of new families of facets for the group problem.
5. Suggests ways in which group-based mixed inequalities could be improved.

Current Research

- Cut Strengthening: continuous variables

- Derivation of other cuts: including CPL_3-extreme inequalities.

Future research

- Automated code for determining the extreme points of $P\Theta_n$.
- Empirical evaluation of resulting cuts.