Branching Rules Revisited

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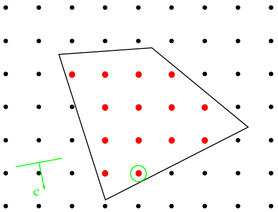
joint work with Tobias Achterberg and Thorsten Koch

Mixed Integer Program (MIP)

$$\begin{aligned} \text{min} \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & l \leq x \leq u \\ & x \in \mathbb{Z}^{n-p} \times \mathbb{R}^p \end{aligned} \qquad \text{(MIP)}$$

with $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, $c, l, u \in \mathbb{R}^n$ and $p \in \{0, \ldots, n\}$.



LP based branch-and-bound

Input: A (MIP)
Output: An opt. solution $x^\star$ or the message "(MIP) is infeasible".

1. Initialize $S := \{P_{\mathsf{LP}}\}$, where P_{LP} is the relaxation of (MIP). Set $c^\star := \infty$.
2. If $S = \emptyset$, exit (return $x^\star$ or "(MIP) is infeasible").
3. Choose a problem $Q \in S$ and delete it from S.
4. Solve the LP $c_Q = \min\{c^T x \mid x \in Q\}$ with opt. solution $\bar{x}$ (Q is possibly strengthened by cuts).
5. If $c_Q \geq c^\star$, goto 2.
6. If $\bar{x}$ integer, set $c^\star := c_Q$ and $x^\star := \bar{x}$, and goto 2.
7. Branching: Split Q into subproblems, add them to S and goto 3.

Branching

Branching = Branching on linear inequalities

(a) Branching on trivial inequalities (= Branching on variables)

- Land & Powel (1979)
- Linderoth & Savelsbergh (1999)
- ...

(b) Branching on non-trivial inequalities

- Clochard & Naddef (1993)
- Borndörfer, Ferreira, Martin (1998)
- Naddef (2002)
- Fischetti, Lodi (2003)
- ...

Variable Selection

Input: Subproblem Q with fractional LP solution $\bar{x}$.
Output: $i \in I$ with $\bar{x}_i \notin \mathbb{Z}$.

1. Let $C = \{i \in I \mid \bar{x}_i \notin \mathbb{Z}\}$ be the set of branching candidates.
2. For all candidates $i \in C$, calculate a score value $s_i \in \mathbb{R}$.
3. Return an index $i \in C$ with $s_i = \max_{j \in C}\{s_j\}$.

Straight Away Strategies

Most Infeasible

Choose variable closest to 0.5, i. e.,

$$s_i = 0.5 - |\bar{x}_i - \lfloor \bar{x}_i \rfloor - 0.5|$$

+ seems to have the most impact on the new LPs.
+ fast to compute

Straight Away Strategies

Random

Choose variable randomly, i. e.,

$$s_i = rand()$$

The whole topic is anyway only about “reading tea leaves”!

Strategy	B&B nodes		time (sec)		fails
	total	geom.	total	geom.	
random	21 246 277	275 789.4	46 202.2	923.6	11
most infeasible	19 421 589	262 368.9	48 037.5	938.0	11

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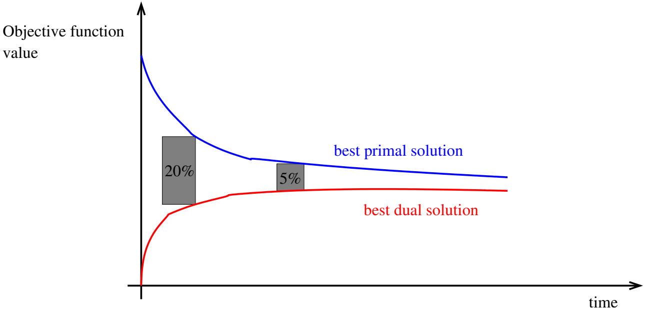
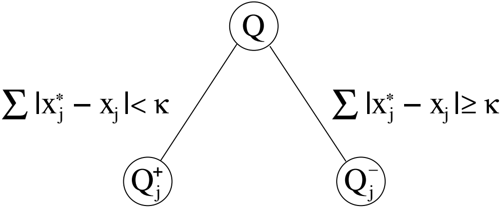
Goals of Branching

1. Improve primal bound
2. Improve dual bound



Primal Branching Rules

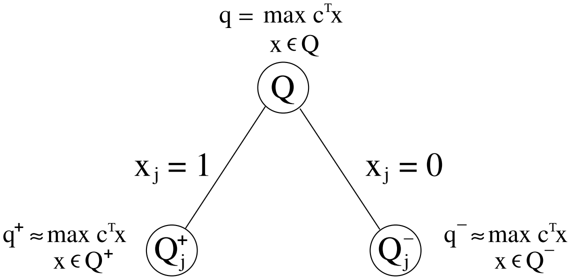
- Local Branching
  Fischetti & Lodi (2003)



- Relaxation Induced Neighborhood Search (RINS)
  Danna, Rothberg & Le Pape (2005)
- Guided Dives
  Danna, Rothberg & Le Pape (2005)

Dual Branching Rules

Measure the success in the increase of the objective function



$$s_i = \text{score}(q^-, q^+) := (1-\mu)\cdot\min\{q^-, q^+\} + \mu\cdot\max\{q^-, q^+\},$$

where μ is some scaling factor, e. g. $\mu = \frac{1}{6}$.

Strong Branching

1. Select $C' \subseteq \{i \mid \bar{x}_i \notin \mathbb{Z}\}$
2. For each $i \in C'$
   (a) Temporally set $u_i = \lfloor \bar{x}_i \rfloor$
   (b) Perform γ simplex iterations yielding obj. fct. value c_i^-
   (c) Temporally set $l_i = \lceil \bar{x}_i \rceil$
   (d) Perform γ simplex iterations yielding obj. fct. value c_i^+
   (e) Set $s_i = \text{score}(c_i^-, c_i^+)$
3. Return an index $i \in C$ with $s_i = \max_{j \in C}\{s_j\}$.

see CPLEX 7.5 and Applegate, Bixby, Chvátal, Cook (2003)

Full Strong Branching

(i) $\gamma = \infty$
(ii) $C' = \{i \mid \bar{x}_i \notin \mathbb{Z}\}$

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Pseudocost Branching

Keep a history of the success of the variables which has already been branched on, see Benichou et al (1971).

$$\begin{aligned}
\varsigma_i^+ &= (\bar{c}_{Q_i^+} - \bar{c}_Q) \,/\, (\lceil \bar{x}_i \rceil - \bar{x}_i) \\
\sigma_i^+ &= \sum_i \varsigma_i^+ \\
\eta_i^+ &= \text{number of these problems solved} \\
\Psi_i^+ &= \sigma_i^+ / \eta_i^+.
\end{aligned}$$

Pseudocost branching

1. Let $C = \{i \in I \mid \bar{x}_i \notin \mathbb{Z}\}$ be the set of candidates.
2. For all candidates $i \in C$, use
   $s_i = \text{score}(\,(\bar{x}_i - \lfloor \bar{x}_i \rfloor) \cdot \Psi_i^-,\ (\lceil \bar{x}_i \rceil - \bar{x}_i) \cdot \Psi_i^+)$
3. Return an index $i \in C$ with $s_i = \max_{j \in C}\{s_j\}$.

Hybrid Strong/Pseudocost Branching

Problem

Uninitialized pseudocosts $\sigma_i^+ = \eta_i^+ = 0$ at the beginning.

Solutions

(1) Initialize pseudocost values with strong branching values
Linderoth & Savelsbergh (1999)

(2) Use strong branching up to level d in the B & B tree,
use pseudocost branching from level $d+1$ on.
see, for instance, LINDO.

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Two simple new ideas

- Use strong branching not only on variables with uninitialized pseudocosts, but also on variables with unreliable pseudocosts.

  The pseudocosts of variable i are called *unreliable*, if

  $$\min\{\eta_i^-, \eta_i^+\} < \eta_{\text{rel}},$$

  with $\eta_{\text{rel}} \in \mathbb{N}$ being the reliability parameter.

- Select the set C of candidates dynamically.

  Introduce a so-called look ahead parameter λ.

  If the best score does not change for λ variables, then stop calling strong branching.

Reliability Branching

1. Let $C = \{i \in I \mid \bar{x}_i \notin \mathbb{Z}\}$ be the set of candidates.
2. Sort C according to non-increasing pseudocosts.

   For all $i \in C$ with $\min\{\eta_i^-, \eta_i^+\} < \eta_{\text{rel}}$, do:

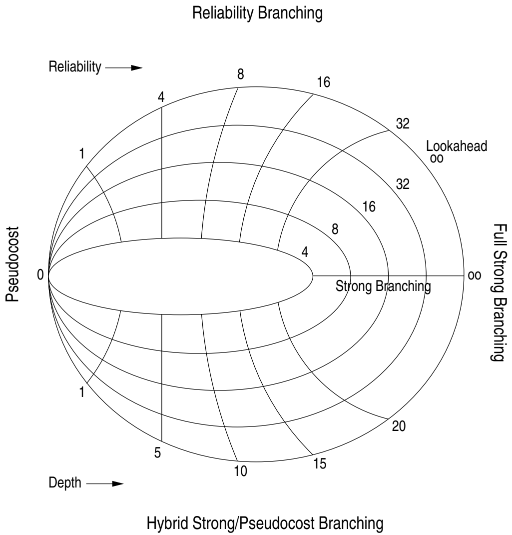
   (a) Perform γ simplex iterations on Q_i^- and Q_i^+.
   Let $\tilde{\Delta}_i^-$ and $\tilde{\Delta}_i^+$ be the objective gains.

   (b) Update the pseudocosts Ψ_i^- and Ψ_i^+ with $\tilde{\Delta}_i^-$ and $\tilde{\Delta}_i^+$.

   (c) Update the score $s_i = \text{score}(\tilde{\Delta}_i^-, \tilde{\Delta}_i^+)$.

   (d) If the maximum score $s^* = \max_{j \in C}\{s_j\}$ has not changed for λ consecutive score updates, goto 3.
3. Return an index $i \in C$ with $s_i = \max_{j \in C}\{s_j\}$.

Branching Rule Classification



Test Set

Instances are taken from

- Miplib 2003, see `http://miplib.zib.de`
- Mittelmann 2003, see `http://plato.asu.edu/bench.html`

where CPLEX 9.0 needs

- at least 5000 B& B nodes
- at most 1 hour CPU time
  (on a 833 MHz Alpha with 4 MB Cache and 2 GB RAM)

These are 24 instances:

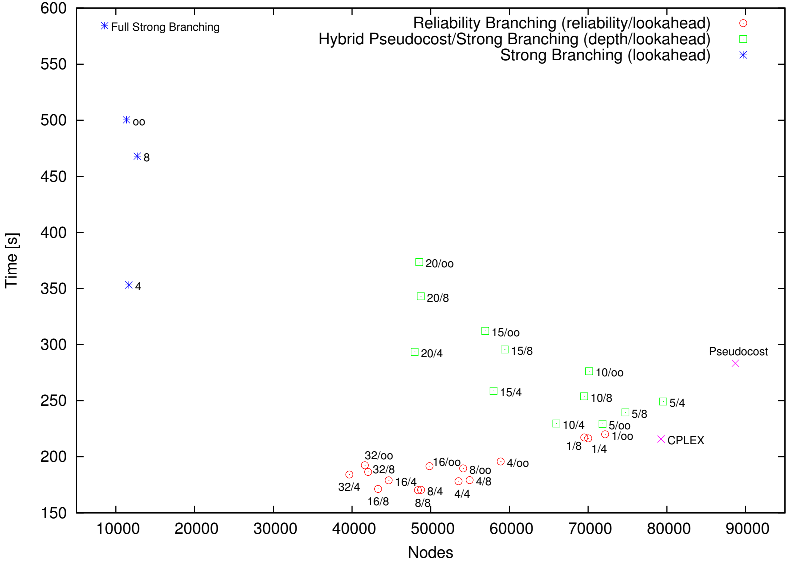
aflow30a	cap6000	gesa2-o	mas74	mas76	misc07
pp08aCUTS	qiu	rout	vpm2	ran8x32	ran10x26
ran13x13	mas284	prod1	bc1	bienst1	neos2
swath1	swath2	neos7	pk1	neos3	ran12x21

Strategy	B&B nodes		time (sec)		strong branchings		fails
	total	geom.	total	geom.	total	geom.	
random	21 246 277	275 789.4	46 202.2	923.6	0	0.0	11
most infeasible	19 421 589	262 368.9	48 037.5	938.0	0	0.0	11
pseudocost	9 381 001	88 706.8	19 945.8	283.4	0	0.0	2
full strong	929 347	7 241.7	26 397.2	504.4	15 569 652	146 784.2	2
lookahead = 4							
strong/pscost (5)	9 698 397	79 535.5	19 487.8	249.3	5 792	216.2	2
strong/pscost (10)	8 251 942	65 966.3	16 499.4	229.6	74 812	2 284.2	2
strong/pscost (15)	7 982 847	57 976.8	17 855.3	258.8	523 377	8 137.8	2
strong/pscost (20)	7 890 374	47 958.5	19 175.6	293.5	2 825 100	17 780.2	2
reliability (1)	9 000 334	70 013.6	17 199.6	216.4	39 126	374.8	1
reliability (4)	6 906 698	53 522.9	13 402.9	178.2	110 628	1 176.7	0
reliability (8)	7 937 968	48 772.8	11 132.7	170.5	117 643	1 850.3	0
reliability (16)	6 022 024	44 649.9	10 782.6	179.0	187 578	3 640.6	0
reliability (32)	7 940 797	39 655.2	11 103.0	184.2	253 014	5 837.8	0
strong branching	1 325 589	11 639.5	20 427.2	353.2	9 965 454	86 188.6	3

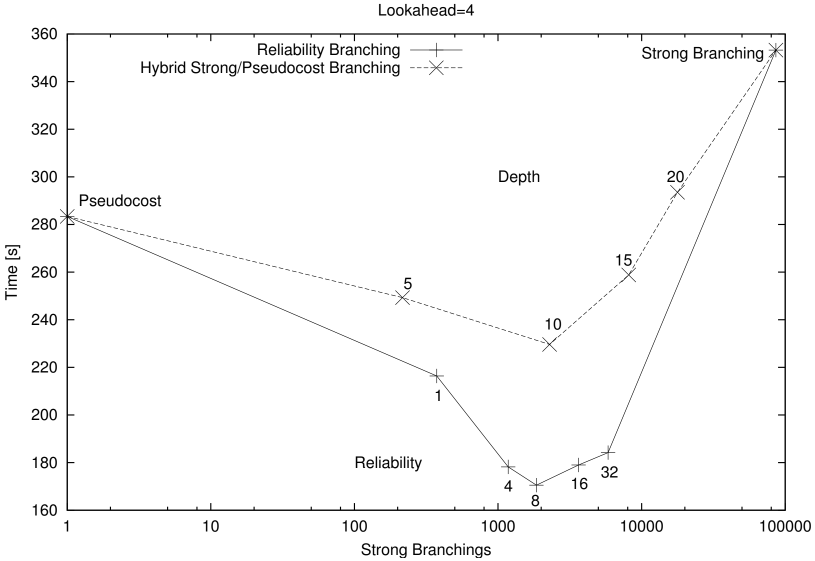
Strategy	B&B nodes		time (sec)		strong branchings		fails
	total	geom.	total	geom.	total	geom.	
lookahead = 8							
strong/pscost (5)	8 653 318	74 730.8	17 389.5	239.6	7 397	268.6	1
strong/pscost (10)	8 332 543	69 489.0	18 362.5	253.9	102 082	2 523.8	2
strong/pscost (15)	7 456 685	59 398.8	20 479.8	295.7	750 577	9 983.1	2
strong/pscost (20)	7 551 419	48 736.5	22 388.8	343.1	3 695 577	20 837.9	3
reliability (1)	8 663 537	69 501.0	16 753.8	217.2	53 557	429.0	1
reliability (4)	8 338 386	54 937.7	12 497.2	179.2	74 906	1 104.6	0
reliability (8)	7 813 409	48 377.3	12 380.2	170.2	133 545	1 998.0	0
reliability (16)	7 579 400	43 311.9	11 946.7	171.3	185 136	3 589.7	0
reliability (32)	7 207 836	42 047.5	11 835.7	186.5	259 482	5 913.2	0
strong branching	1 294 569	12 714.7	25 619.4	468.0	12 651 504	119 799.1	4

Strategy	B&B nodes total	B&B nodes geom.	time (sec) total	time (sec) geom.	strong branchings total	strong branchings geom.	fails
lookahead = ∞							
strong/pscost (5)	8 498 292	71 817.4	18 116.9	229.3	14 675	489.9	2
strong/pscost (10)	9 247 636	70 125.8	20 472.1	276.2	154 458	3 870.1	2
strong/pscost (15)	6 670 440	56 926.6	19 907.4	312.2	890 187	13 127.2	3
strong/pscost (20)	7 627 640	48 547.0	23 538.5	373.6	3 842 516	26 557.4	3
reliability (1)	7 747 290	72 159.1	15 825.8	220.0	48 162	408.4	1
reliability (4)	9 068 723	58 886.4	14 258.1	195.8	78 625	1 096.6	2
reliability (8)	8 551 045	54 118.3	13 563.0	189.6	135 541	2 042.9	1
reliability (16)	6 567 432	49 839.9	12 766.4	191.6	196 220	3 601.0	0
reliability (32)	7 502 942	41 636.1	12 393.6	192.5	281 822	6 000.7	0
strong branching	1 163 822	11 355.7	26 176.4	500.3	12 793 364	127 737.1	4
CPLEX/SIP cuts	10 467 429	79 269.0	18 617.4	215.8	—	—	1

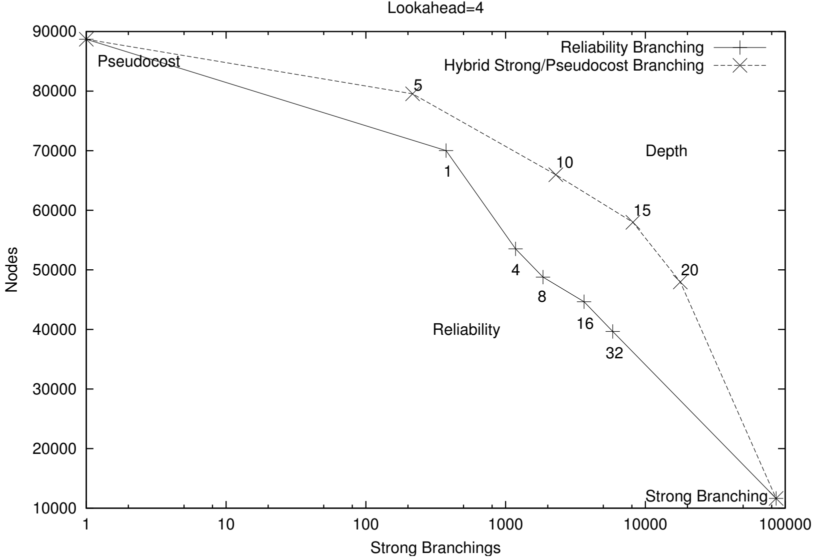
Nodes versus Time



Time versus Strong Branchings



Nodes versus Strong Branchings



Conclusions

Summary

- most infeasible as good as random
- *strong branching* is best with respect to number of nodes, but not with respect to time
- *reliability branching* outperforms *hybrid strong/pseudocost branching*
- Increasing η_{rel} (or the depth d) decreases the number of nodes
- Currently best choice $\eta_{\text{rel}} = 8$ and $\lambda = 4$.

Open

- Bridge the gap to *full strong branching* without increasing the running time.
- Missing Theory !?
- Branching versus modeling with binary variables.