MIP models for MIP separation

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MIPping crucial parts of the MIP solver

- It is well known that the solution of very hard MIPs can take advantage of the solution of a series of **auxiliary LPs** intended to guide the main steps of the MIP solver.

  E.g.: LP models used to control the branching strategy (**strong branching**), the cut generation (**lift-and-project**), the primal heuristics (**reduced costs**), etc.

- Also well known is the fact that finding **good heuristic MIP solutions** often requires a computing time that is just comparable to that needed to solve the LP relaxation of the problem at hand.

- This leads to the idea of "**translating into a MIP model**" (**MIPping**) some crucial decisions to be taken within a MIP algorithm (How to improve the incumbent solution? How to branch? **How to cut?**), with the aim of bringing the MIP technology well within the MIP solver.

- Very recently, the MIPping approach has been applied by several authors to modeling and solving (possibly in a heuristic way) the NP-hard separation problems of famous classes of valid inequalities for mixed integer linear programs.

MIPping Chvátal-Gomory (fractional) cuts

- Consider first the pure integer linear programming problem

$$\min\{c^T x : Ax \leq b, x \geq 0, x \text{ integral}\}$$

where A is an $m \times n$ matrix, along with the two associated polyhedra

$$P := \{x \in \mathbb{R}^n_+ : Ax \leq b\}$$

$$P_I := conv\{x \in \mathbb{Z}^n_+ : Ax \leq b\} = conv(P \cap \mathbb{Z}^n)$$

- **A Chvátal-Gomory (CG) cut** (also known as *Gomory fractional* cut) is an inequality of the form

$$\lfloor u^T A \rfloor x \leq \lfloor u^T b \rfloor$$

where $u \in \mathbb{R}^m_+$ is a vector of nonnegative multipliers, and $\lfloor \cdot \rfloor$ denotes the lower integer part.

- The **Chvátal closure** of P is defined as

$$P^1 := \{x \geq 0 : Ax \leq b, \lfloor u^T A \rfloor x \leq \lfloor u^T b \rfloor \text{ for all } u \in \mathbb{R}^m_+\}. \quad (1)$$

- By the well-known equivalence between optimization and separation, optimizing over the first Chvátal closure is equivalent to solving the following NP-hard (Eisenbrand, 1999) separation problem:

  CG-SEP: Given any point $x^* \in P$ find (if any) a CG cut $\alpha^T x \leq \alpha_0$ that is violated by x^*, i.e., find $u \in \mathbb{R}^m_+$ such that $\lfloor u^T A \rfloor x^* > \lfloor u^T b \rfloor$, or prove that no such u exists.

- CG-SEP can be **MIPped** as follows (Fischetti and Lodi, 2005): Let $\alpha = \lfloor u^T A \rfloor$ and $\alpha_0 = \lfloor u^T b \rfloor$ for some (unknown) $u \in \mathbb{R}^m_+$.

$$\max \quad \alpha^T x^* - \alpha_0 \tag{2}$$

$$\alpha^T \leq u^T A, \; u \geq 0 \tag{3}$$

$$\alpha_0 + 0.9999 \geq u^T b \tag{4}$$

$$\alpha, \; \alpha_0 \quad \text{integer} \tag{5}$$

- Validity of model (2)-(5) follows from the fact that $\alpha^T x \leq \alpha_0$ is a CG cut if and only if (α, α_0) is an integral vector, as stated in (5), and $\alpha^T x \leq \alpha_0 + 0.9999$ is a valid inequality for P, as stated in (3)-(4) by Farkas' lemma.

MIPping projected Chvátal-Gomory cuts

- Bonami, Cornuéjols, Dash, Fischetti and Lodi (2005) addressed the extension of Chvátal-Gomory cuts to the **mixed-integer case**:

$$\min\{c^T x + f^T y : Ax + Cy \leq b, x \geq 0, x \text{ integral}, \ y \geq 0\} \tag{6}$$

where A and C are $m \times n$ and $m \times r$ matrices, respectively. Let

$$P(x,y) := \{(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^r_+ : Ax + Cy \leq b\} \tag{7}$$

$$P_I(x,y) := conv(\{(x,y) \in P(x,y) : x \text{ integral}\}). \tag{8}$$

- Define the **projection** of $P(x,y)$ onto the space of the x variables as:

$$P(x) \quad := \{x \in \mathbb{R}^n_+ : \text{there exists } y \in \mathbb{R}^r_+ \text{ s.t. } Ax + Cy \leq b\} \tag{9}$$

$$= \{x \in \mathbb{R}^n_+ : u^k A \leq u^k b, \ k = 1, \ldots, K\} \tag{10}$$

$$=: \{x \in \mathbb{R}^n_+ : \bar{A}x \leq \bar{b}\} \tag{11}$$

where $u^1, \ldots, u^K$ are the (finitely many) extreme rays of the projection cone $\{u \in \mathbb{R}^m_+ : u^T C \geq 0^T\}$.

- We define a **projected Chvátal-Gomory (pro-CG) cut** as a CG cut derived from the system $\bar{A}x \leq \bar{b},\ x \geq 0$, i.e., an inequality of the form $\lfloor w^T \bar{A} \rfloor x \leq \lfloor w^T \bar{b} \rfloor$ for some $w \geq 0$.

- Any row of $\bar{A}x \leq \bar{b}$ can be obtained as a linear combination of the rows of $Ax \leq b$ with multipliers $\bar{u} \geq 0$ such that $\bar{u}^T C \geq 0^T$ $\Rightarrow$ a pro-CG cut can equivalently (and more directly) be defined as an inequality of the form:

$$\lfloor u^T A \rfloor x \leq \lfloor u^T b \rfloor \qquad \text{for any } u \geq 0 \text{ such that } u^T C \geq 0^T. \tag{12}$$

- As a consequence, the associated separation problem can be MIPped as a **simple extension** of its CG counterpart (2)-(5):

$$\max \quad \alpha^T x^* - \alpha_0 \tag{13}$$

$$\alpha^T \leq u^T A \tag{14}$$

$$0^T \leq u^T C \tag{15}$$

$$\alpha_0 + 0.9999 \geq u^T b \tag{16}$$

$$u \geq 0 \tag{17}$$

$$\alpha,\ \alpha_0 \quad \text{integer} \tag{18}$$

MIPping split cuts

- Consider a generic MIP associated with the polyhedra

$$P(x,y) := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^r : Ax + Cy \geq b\} \tag{19}$$

$$P_I(x,y) := conv(\{(x,y) \in P(x,y) : x \text{ integral}\}). \tag{20}$$

where the variable bounds (if any) are included among the explicit constraints.

- For any $\pi \in \mathbb{Z}^n$ and $\pi_0 \in \mathbb{Z}$, the **disjunction** $\pi^T x \leq \pi_0$ or $\pi^T x \geq \pi_0 + 1$ is valid for $P_I(x,y) \Rightarrow P_I(x,y) \subseteq conv(\Pi_0 \cup \Pi_1)$ where

$$\Pi_0 := P(x,y) \cap \{(x,y) : -\pi^T x \geq -\pi_0\} \tag{21}$$

$$\Pi_1 := P(x,y) \cap \{(x,y) : \pi^T x \geq \pi_0 + 1\}. \tag{22}$$

- A valid inequality $\alpha^T x + \gamma^T y \geq \beta$ for $conv(\Pi_0 \cup \Pi_1)$ is called a **split cut** (Cook, Kannan and Schrijver, 1990).

- The convex set obtained by intersecting $P(x, y)$ with all the split cuts is called the **split closure** of $P(x, y)$. Optimization over the slit closure in **NP-hard** (Caprara and Letchford, 2003).

- Balas and Saxena (2005) MIPped the separation problem for the most violated split cut through the following **(nonlinear) MIP**:

$$\min \quad \alpha^T x^* + \gamma^T y^* - \beta \tag{23}$$

$$\alpha^T = u^T A - u_0 \pi^T, \ \gamma^T = u^T C, \ \beta = u^T b - u_0 \pi_0, (u, u_0) \geq 0 \tag{24}$$

$$\alpha^T = v^T A + v_0 \pi^T, \ \gamma^T = v^T C, \ \beta = v^T b + v_0(\pi_0 + 1), \ (v, v_0) \geq 0 \tag{25}$$

$$u_0 + v_0 = 1 \tag{26}$$

$$\pi, \ \pi_0 \ \text{integer} \tag{27}$$

where
(24) $\Rightarrow$ Farkas' condition for validity of $\alpha^T x + \gamma^T y \geq \beta$ w.r.t.

$$\Pi_0 := P(x, y) \cap \{(x, y) : -\pi^T x \geq -\pi_0\}$$

(25) $\Rightarrow$ Farkas' condition for validity of $\alpha^T x + \gamma^T y \geq \beta$ w.r.t.

$$\Pi_1 := P(x, y) \cap \{(x, y) : \pi^T x \geq \pi_0 + 1$$

- Normalization constraint $u_0 + v_0 = 1$ allows one to simplify the model to:

$$
\begin{aligned}
\min u^T(Ax^* + Cy^* - b) \quad &- \quad u_0(\pi^T x^* - \pi_0) && (28)\\
u^T A - v^T A - \pi \quad &= \quad 0 && (29)\\
u^T C - v^T C \quad &= \quad 0 && (30)\\
-u^T b + v^T b + \pi_0 \quad &= \quad u_0 - 1 && (31)\\
0 < u_0 < 1 \quad &, \quad u, v \geq 0 && (32)\\
\pi,\ \pi_0 \quad & \quad \text{integer} && (33)
\end{aligned}
$$

- Note that the **nonlinearity** only arises in the objective function $\Rightarrow$ for any fixed value of parameter u_0 the model becomes a regular MIP (w.l.o.g. $u_0 \in (0, 1/2]$)

- Balas and Saxena considered a **heuristic list** of possible values for parameter u_0, say $(0.05, 0.1, 0.2, 0.3, 0.4, 0.5)$ and then enriched it, on the fly, by inserting new heuristic points.

MIPping MIR cuts

- Dash, Günlük and Lodi (2005) addressed the optimization over the split closure by looking at its equivalent definition in terms of **MIR inequalities**. Let

$$P(x,y) = \{(x,y) \in R^n_+ \times R^r_+ \ : \ Ax + Cy + Is = b, \ s \geq 0\} \quad (34)$$

- Find a violated MIR inequality for a given (x^*, y^*, s^*) $\Rightarrow$ define an **"approved by Farkas"** integer vector $(\bar{\alpha}, \bar{\beta})$ and a nonnegative vector $(u^+, \hat{\alpha}, \hat{\gamma}, \hat{\beta}, \Delta)$ with $0 < \hat{\beta} < 1$ and $0 < \Delta < 1$ such that

$$u^+ s^* + \hat{\gamma}^T y^* + \hat{\alpha}^T x^* < \hat{\beta}\Delta \quad (35)$$

- The RHS can be **linearized** (in an approximate way) as follows. We first approximate the unknown $\hat{\beta} \in (0,1)$ by $\hat{\beta} \approx \sum_{k=1}^{K} 2^{-k}\pi_k$ for unknown binary variables π_k's.

- The nonlinear RHS is then approximated by

$$\hat{\beta}\Delta \approx \Delta \sum_{k=1}^{K} 2^{-k}\pi_k = \sum_{k=1}^{K} 2^{-k}\Phi_k,$$

where the terms $\Phi_k := \Delta\pi_k$ can be linearized easily (because π is binary and $0 \leq \Delta < 1$).

- Using this approach, Dash, Günlük and Lodi **MIPped** MIR separation as follows:

$$\min u^+ s^* + \hat{\gamma}^T y^* + \hat{\alpha}^T x^* - \sum_{k=1}^{K} 2^{-k}\Phi_k \tag{36}$$

$$\hat{\gamma} \geq u^T C \tag{37}$$

$$\hat{\alpha}^T + \bar{\alpha}^T \geq u^T A \tag{38}$$

$$\hat{\beta} + \bar{\beta} \leq u^T b \tag{39}$$

$$u^+ \geq u \tag{40}$$

$$u^+,\ \hat{\alpha},\ \hat{\beta}, \hat{\gamma} \geq 0, \quad (\bar{\alpha}, \bar{\beta}) \text{ integer} \tag{41}$$

$$\Delta = (\bar{\beta} + 1) - \bar{\alpha}^T x^* \tag{42}$$

$$\hat{\beta} = \sum_{k=1}^{K} 2^{-k}\pi_k \tag{43}$$

$$\Phi_k \leq \Delta \quad \text{for all } k = 1, \ldots, K \tag{44}$$

$$\Phi_k \leq \pi_k \quad \text{for all } k = 1, \ldots, K \tag{45}$$

$$\pi_k \in \{0, 1\} \quad \text{for all } k = 1, \ldots, K \tag{46}$$

(47)

Strengthen of the closures

- The strengthen of the closures, namely CG, pro-CG and split (or MIR) closures, has been evaluated by running a cutting plane algorithm for a large (sometimes huge) computing time.

- **Goal of the investigation**: show the tightness of the closures, rather than investigating the practical relevance of the separation MIPping idea when used within a practical MIP solver.

- Tightness of the closures for MIPlib 3.0 instances, in terms of "**percentage of gap closed**", computed as

$$100 - 100(opt\_value(P_I) - opt\_value(P^1))/(opt\_value(P_I) - opt\_value(P))$$

		Split closure	CG closure
% Gap closed	Average	71.71	62.59
% Gap closed	98-100	9 instances	9 instances
% Gap closed	75-98	4 instances	2 instances
% Gap closed	25-75	6 instances	7 instances
% Gap closed	< 25	6 instances	7 instances

Table 1: Percentage of gap closed for 25 *pure* integer linear programs in the MIPlib 3.0.

		Split closure	pro-CG closure
% Gap closed	Average	84.34	36.38
% Gap closed	98-100	16 instances	3 instances
% Gap closed	75-98	10 instances	3 instances
% Gap closed	25-75	2 instances	11 instances
% Gap closed	< 25	5 instances	17 instances

Table 2: Percentage of gap closed for 33 *mixed* integer linear programs in the MIPlib 3.0.

On the practical relevance of rank-1 cuts

- **Lesson learned**: in most practical cases the inequalities of rank 1 already give a **very tight approximation** of the convex hull of integer and mixed-integer programs.

- Nice features of the rank-1 cuts separated through the MIPping approach: **very sparse** and **numerically stable** (as opposed to, e.g., GMI cuts read from the tableau)

Pure IP Instance	# Int Variables	Mean Support Size
nw04	87482	2.084
air05	7195	8.210
seymour	1372	5.263
misc03	159	3.771
p0033	33	4.847

Mixed IP Instance	# Int Variables	Mean Support Size
qnet1_o	1417	6.690
gesa2_o	720	4.937
arki001	538	3.146
vpm1	168	4.503
pp08aCUTS	64	3.850

Figure 1: Split cut properties (from Balas and Saxena, 2005)

Why are rank-1 CG/split cuts so nice?

Note that the notion of rank depends on the **formulation**: rank-2 cuts are just rank-1 cuts for a different formulation!

a. The original formulation has a **nice structure** (sparse constraints, small coefficients, small basis determinants, etc.) that naturally produces nice rank-1 CG/split cuts—this property deteriorates as soon as new cuts are added to the formulation...

b. There is nothing special with rank-1 cuts, the nice behavior derives from the MIPping approach that solves an **optimization problem** within the separation procedure, i.e., we make a **clever choice** of the separated cut within a large family...

c. **...**

MIPing drawbacks

- As implemented, the MIPping approach is **exceedingly time consuming** ...
- However, one can easily implement a **hybrid approach** in which the MIP-based separation procedures are applied (for a fixed amount of time) in a **preprocessing** phase, resulting in a tighter MIP formulation to be solved at a later time by a standard MIP solver.
- Using this idea, two **unsolved MIPlib-2003** instances, namely **nsrand-ipx** and **arki001**, have been solved to proven optimality for the first time by Fischetti and Lodi (2005) and by Balas and Saxena (2005)
- For very difficult and challenging problems it does pay to invest a large amount of computing time to **improve the formulation** by adding cuts in these closures before switching to either general- or special-purpose solution algorithms.

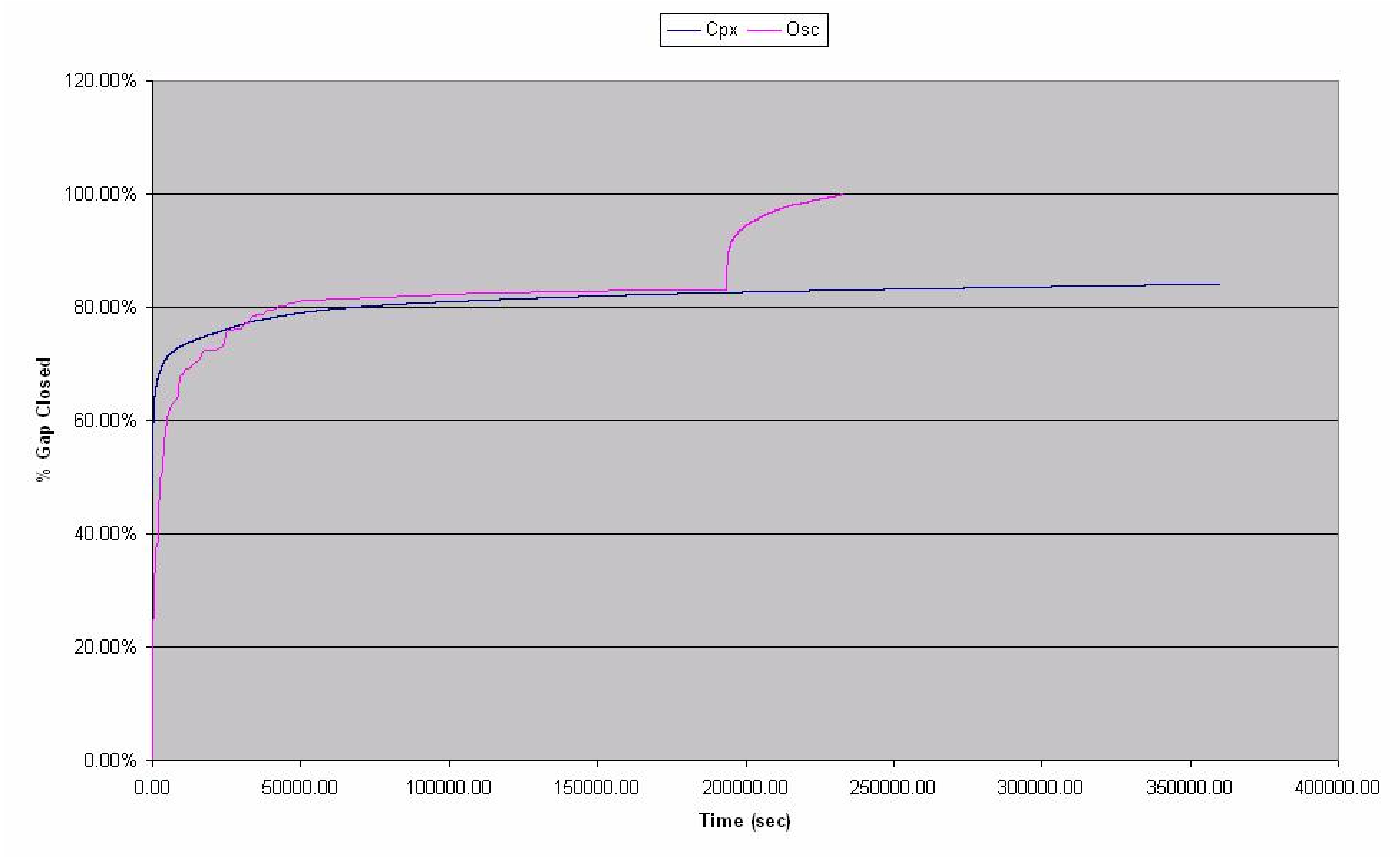


Figure 2: Split cut preprocessing vs. enumeration (from Balas and Saxena, 2005)

Future directions of work on split cuts

- Read and use information from the optimal **tableau rows** to have a **warm start** for the MIP solver—viewed as sort of "clean up" procedure for standard GMI cuts, potentially stronger than their **lift-and-projec**t and **reduce-and-split** counterparts...

- Avoid at all the **explicit definition** of the MIP separation model $\Rightarrow$ **design very fast ad-hoc heuristics** that use the underlying MIP separation model only implicitly...

- Concentrate on a suitable **subclass of split cuts**...

  E.g., one can address the split cuts associated with **any binary disjunction** $\pi x \leq \pi_0$ or $\pi x \geq \pi_0 + 1$, $(\pi, \pi_0) \in \{0, 1\}^{n+1}$ $\Rightarrow$ these cuts generalize lift-and-project cuts and are easily MIPped through a (linear) 0-1 MIP

Split cuts over cardinality disjunctions

joint work with A. Tramontani (in progress)

- Given the pure ILP

$$\min\{c^T x : Ax \geq A_0, x \geq 0 \text{ integer}\} \tag{48}$$

and two valid inequalities for the LP relaxation

$$a^T x \geq a_0 \quad \text{and} \quad b^T x \geq b_0 \quad \text{with} \quad \Delta := b_0 - a_0 \geq 0,$$

the **pairing inequality** $\gamma^T x \geq \gamma_0 := b_0$ is valid for the integer feasible points (Günlük and Pochet, 2001), where

$$\gamma_j := \begin{cases} a_j, & \text{if } a_j \geq b_j \\ b_j, & \text{if } a_j \leq b_j \text{ and } b_j \leq a_j + \Delta \\ a_j + \Delta, & \text{if } a_j \leq b_j \text{ and } b_j \geq a_j + \Delta. \end{cases} \tag{49}$$

- Re-writing

$$\gamma_j := \max\{a_j, \min\{a_j + \Delta, b_j\}\}, \qquad j = 1, \ldots, n$$

the pairing separation problem for a given x^* can be MIPed as:

$$\max \quad b_0 - \gamma^T x^* \tag{50}$$

$$a^T \geq u^T A_0, \quad a_0 \leq u^T A_0 \tag{51}$$

$$b^T \geq v^T A_0, \quad b_0 \leq v^T A_0 \tag{52}$$

$$\Delta = b_0 - a_0 \geq 0 \tag{53}$$

$$\gamma \geq a, \quad \gamma \geq b - M\theta, \quad \gamma \geq a + \Delta_n - M(1 - \theta) \tag{54}$$

$$a + \Delta_n \geq b - M\theta, \quad a + \Delta_n \leq b + M(1 - \theta) \tag{55}$$

$$0 \leq u \leq 1, \quad 0 \leq v \leq 1 \tag{56}$$

$$\theta \in \{0, 1\}^n \tag{57}$$

where $\Delta_n := (\Delta, \Delta, \ldots, \Delta)$ and M is a large positive value.

- By elaborating the above model it is not difficult to get rid of the big-M terms and to show that **pairing inequalities are equivalent to split cuts defined over the cardinality disjunctions**

$$\theta^T x \leq \theta_0 \quad \text{or} \quad \theta^T x \geq \theta_0 + 1. \tag{58}$$

for $\theta \in \{0,1\}^n$ and $\theta_0 = 0$.

- The pairing inequality separation problem can therefore be reformulated as in Balas and Saxena (2005) thorough the nonlinear MIP:

$$
\begin{aligned}
\max \quad & \gamma_0 - \gamma^T x^* && (59)\\
& \gamma^T \geq u^T A - u_0 \theta^T && (60)\\
& \gamma^T \geq v^T A + v_0 \theta^T && (61)\\
& \gamma_0 = u^T A_0 && (62)\\
& \gamma_0 = v^T A_0 + v_0, && (63)\\
& u \geq 0, \quad v \geq 0, \quad 0 \leq u_0 \leq 1, \quad 0 \leq v_0 \leq 1 && (64)\\
& \theta \in \{0,1\}^n
\end{aligned}
$$

- By imposing w.l.o.g. $u_0 = 1$, $\gamma^T = u^T A - u_0 \theta^T + w^T$, $w \geq 0$, we get

$$\max \quad \theta^T x^* - u^T(Ax^* - A_0) - w^T x^* \tag{65}$$
$$u^T A - v^T A - (1 + v_0)\theta^T + w^T \geq 0 \tag{66}$$
$$u^T A_0 - v^T A_0 - v_0 = 0 \tag{67}$$
$$u \geq 0, \quad v \geq 0, \quad w \geq 0, \quad 0 \leq v_0 \leq 1 \tag{68}$$
$$\theta \in \{0, 1\}^n$$

that can be linearized easily by introducing continuous variables α_j to replace the nonlinear terms $v_0 \theta_j$. This leads to the following **0-1 MIP**:

$$\max \quad \theta^T x^* - u^T(Ax^* - A_0) - w^T x^*$$
$$u^T A - v^T A - \theta^T - \alpha^T + w^T \geq 0 \tag{69}$$
$$u^T A_0 - v^T A_0 - v_0 = 0$$
$$\alpha_j \geq v_0 - 1 + \theta_j \qquad \text{for all } j = 1, \ldots, n \tag{70}$$
$$u \geq 0, \quad v \geq 0, \quad w \geq 0, \quad \alpha \geq 0, \quad 0 \leq v_0 \leq 1 \tag{71}$$
$$\theta \in \{0, 1\}^n$$

Heuristic strengthening of cardinality split cuts

- Given a solution (u, v, v_0) of the above model, a simple **heuristic procedure for strengthening** the cut $\gamma^T x \geq \gamma_0$ is as follows:

1. **compute the optimal disjunction** $\pi^T x \leq 0$ or $\pi^T x \geq 1$ ($\pi \in \mathcal{Z}^n$) for the given multipliers (u, v, v_0), by using the Balas-Jeroslow procedure;

2. given the disjunction corresponding to the above π, **compute the "optimal" multipliers** $(\tilde{u}, \tilde{u}_0, \tilde{w})$ by solving the LP

$$\max \quad u_0 \pi^T x^* - u^T s^* - w^T t^* \tag{72}$$

$$u^T A - v^T A - u_0 \pi^T - v_0 \pi^T + w^T \geq 0 \tag{73}$$

$$u^T A_0 - v^T A_0 - v_0 = 0 \tag{74}$$

$$\sum u_i + \sum v_i + \sum w_j + u_0 + v_0 = 2m + n + 2 \tag{75}$$

$$u \geq 0, \quad v \geq 0, \quad w \geq 0, \quad u_0 \geq 0, \quad v_0 \geq 0 \tag{76}$$

where $s^* := \max\{\epsilon, Ax^* - A_0\}$ and $t^* := \max\{\epsilon, x^*\}$ ($\epsilon = 10^{-6}$);

3. compute the cut $\gamma^T x \geq \gamma_0$ as $\gamma^T := \tilde{u}^T A - \tilde{u}_0 \pi^T + \tilde{w}$, $\gamma_0 := \tilde{u}^T A_0$.

Instance	L&P gap-c%	CD gap-c%	CD++ gap-c%	Split gap-c%
air03	100.00	100.00	100.00	100.00
air04	32.61	31.67	45.94	62.42
air05	36.05	29.83	39.06	62.05
cap6000	46.67	63.79	63.79	37.63
fiber	22.40	92.26	98.47	98.50
harp2	4.13	3.38	24.23	17.50
l152lav	34.46	60.34	84.82	92.10
lseu	5.09	20.06	70.70	93.75
misc03	40.21	47.87	48.84	51.47
misc07	11.44	11.46	13.30	19.48
mitre	59.73	10.95	93.64	100.00
mod008	9.02	25.75	26.99	100.00
mod010	52.49	100.00	100.00	100.00
nw04	39.19	100.00	100.00	100.00
p0033	8.19	70.30	70.30	87.42
p0201	46.85	73.62	74.93	74.93
p0282	93.66	96.73	97.32	99.90
p0548	87.11	92.08	96.29	100.00
p2756	86.77	88.06	99.81	92.32
seymour	9.14	2.21	12.06	61.94
stein27	0.00	0.00	0.00	0.00
stein45	0.00	0.00	0.00	0.00
Average gap%	37.51	50.93	61.84	70.52

Table 3: Comparison of the **percentage of gap closed** (gap-c%) within 1,800 CPU sec.s by **Lift and Project** (L&P), **Cardinality Disjunction** (CD), and **Strengthened Cardinality Disjunction** (CD++). Column `Split` is from Balas and Saxena (2005) and refers to **Split cuts** (time-limit for *each* separation call: 3,600 sec.s). Anomalies for `cap6000`, `harp2` and `p2756` due to different pre-processing.

	Cardinality Disjunction				Strength. Cardinality Disjunction			
Instance	n.iter	n.cuts	gap-c%	time (sec.s)	n.iter	n.cuts	gap-c%	time (sec.s)
air03	7	21	100.00	0.9	3	4	100.00	0.2
air04	188	1432	31.67	1800.0	140	864	45.94	1800.0
air05	265	1227	29.83	1800.0	229	1085	39.06	1800.0
cap6000	72	149	63.79	10.6	61	135	63.79	12.9
fiber	668	3260	92.26	1800.0	294	862	98.47	1008.3
harp2	1547	7282	3.38	870.7	17	101	24.23	4.2
l152lav	360	1425	60.34	1800.0	706	1482	84.82	1800.0
lseu	29	96	20.06	2.9	23	65	70.70	1.6
misc03	185	450	47.87	1800.0	212	404	48.84	1800.0
misc07	255	777	11.46	1800.0	311	612	13.30	1800.0
mitre	248	680	10.95	1800.0	212	405	93.64	1800.0
mod008	36	123	25.75	2.3	20	66	26.99	0.7
mod010	89	322	100.00	60.9	13	30	100.00	1.2
nw04	112	731	100.00	254.4	28	129	100.00	11.1
p0033	36	92	70.30	0.9	18	45	70.30	0.6
p0201	412	863	73.62	1800.0	202	357	74.93	441.3
p0282	255	1074	96.73	114.5	182	769	97.32	65.9
p0548	725	1826	92.08	1800.0	616	1560	96.29	1800.0
p2756	327	1548	88.06	444.4	959	9048	99.81	1800.0
seymour	52	177	2.21	1800.0	48	141	12.06	1800.0
stein27	255	933	0.00	124.5	32	86	0.00	8.1
stein45	544	2352	0.00	1230.5	164	487	0.00	201.9

Table 4: Comparison between **Cardinality Disjunction** (CD) and **Strengthened Cardinality Disjunction** (SCD).

IPCO 2007 meeting at Cornell University (Ithaca), June 25-27, 2007

- **Program Committee:**

  Dimitris Bertsimas

  Dan Bienstock

  Alberto Caprara

  Bill Cook

  Gerard Cornuejols

  Matteo Fischetti (chair)

  Bertrand Guenin

  Christoph Helmberg

  Tibor Jordn

  Tom McCormick

  David Williamson (organizer)

  Gerhard Woeginger

- **Strong MIP papers strongly encouraged!**

A look beyond disjunctive closures: the KP closure

- According to the recent computational analysis reported in

  M. Fischetti and M. Monaci, How tight is the corner relaxation?, Technical Report, 2005

  the Gomory's corner relaxation gives a **very good approximation** of the integer hull for MIPs with general-integer variables, but...

- ... the approximation is **less effective for problems with 0-1 variables only**, as observed already in

  E. Balas, A Note on the Group-Theoretic Approach to Integer Programming and the 0-1 Case, Operations Research 21, 1, 321-322 (1973).

- **Explanation:** for 0-1 ILPs, even the non-binding variable bound constraints $x_j \geq 0$ or $x_j \leq 1$ play an important role, hence their relaxation produces weaker bounds...

- **How can we take the variable bound constraints $0 \le x_j \le 1$ into account when generating Gomory-like cuts?**

- We introduce the concept of **knapsack closure** as a tightening of the classical Chavtal-Gomory (CG) concept:

  for **all** inequalities $w^T x \le w_0$ valid for the LP relaxation ...

  ... add to the original system **all** the valid inequalities for the knapsack polytope

$$conv\{x \in \{0,1\}^n : w^T x \le w_0\}$$

- **Question:** Is the knapsack closure **significantly tighter** than the classical CG closure?

- Answer (work in progress): actually **optimize** over the KP closure on a significant set of MIPLIB test instances.

The basic machinery

- We are interested in the 0-1 ILP

$$\min\{c^T x : x \in P \cap X\} \tag{77}$$

where

$$P := \{x \in \Re^n : Ax \le b, x \ge 0\} \tag{78}$$

is a given polyhedron and

$$X \subseteq Z^n$$

is such that the optimization of every linear function over X is a "practically tractable" problem, e.g.,

$$X := \{x \in Z^n : 0 \le x \le 1\} \tag{79}$$

- Let $w^T x \leq w_0$ be any valid inequality for P, called **source KP inequality** in the sequel,

  and let
$$KP(w, w_0) := \{x \in X : w^T x \leq w_0\} \tag{80}$$
  define a corresponding **KP relaxation** of the original ILP problem.

- Given a (fractional) point $x^* \in \Re^n$, we are interested in the following

  Separation problem: **Find a linear inequality $\alpha^T x \leq \alpha_0$ that is valid for $KP(w, w_0)$ but violated by x^* (if any)**.

The "easy" case: the source KP inequality is given

- If the source KP inequality is given, the separation problem amounts to the solution of a series of knapsack problems, i.e., of optimizations of a linear function over the KP relaxation $KP(w, w_0)$.

- Indeed, one can in principle enumerate all the members of $KP(w, w_0)$, say $x^1, \ldots, x^K$, and write the following LP model for the separation:

$$\max \quad \alpha^T x^* - \alpha_0 \tag{81}$$

$$\alpha^T x^i \leq \alpha_0, \quad \text{for all } i = 1, \ldots, K \tag{82}$$

$$-1 \leq \alpha_j \leq 1, \quad \text{for all } j = 0, \ldots, n \tag{83}$$

where (83) are just normalization conditions.

- The above LP contains an exponential number of constraints $\Rightarrow$ standard run-time cut generation technique, where at each iteration the following steps are performed:

- consider explicitly **just a few** solutions in $KP(w, w_0)$, say solutions $x^1, \cdots, x^h$ for some $h \ll K$ (initially, $h := 0$)

- compute an optimal solution (α^*, α_0^*) of the corresponding restricted LP model

$$\max \quad \alpha^T x^* - \alpha_0 \tag{84}$$

$$\alpha^T x^i \leq \alpha_0, \quad \text{for all } i = 1, \ldots, h \tag{85}$$

$$-1 \leq \alpha_j \leq 1, \quad \text{for all } j = 0, \ldots, n \tag{86}$$

- if $\alpha^* x^* - \alpha_0^* \leq 0$, then the method can be stopped as no violated inequality $\alpha^T x \leq \alpha_0$ exists

- call an *oracle* to compute an optimal solution y^* of the KP problem

$$\max\{\alpha^* y : y \in KP(w, w_0)\}$$

- if $\alpha^* y^* \leq \alpha_0^*$, then the inequality $\alpha^* x \leq \alpha_0^*$ is valid for $KP(w, w_0)$ and maximally violated, so stop

- include y^* in the separation model by setting $h := h + 1$ and $x^h := y^*$, and repeat.

The "hard" case: the source KP inequality is not given

- We need to extend the method above to the case where the inequality $w^T x \leq w_0$ is **not given** a priori (nor read from the optimal LP tableau etc.), but is completely general and defined during the separation phase so as to maximize its effectiveness.

- This approach produces a much more powerful separation tool that goes far **beyond the separation over the first Chvátal closure**...

  ... but requires to use Farkas' Lemma to certify the validity of $w^T x \leq w_0$ for P, and a more involved MIP model to replace the "easy" LP separation model shown above.

- Here is how the MIP separation model looks like:

$$\max \quad \alpha^T x^* - \alpha_0 \tag{87}$$

$$w^T \le u^T A,\ w_0 \ge u^T b,\ u \ge 0 \tag{88}$$

$$\alpha^T x^i \le \alpha_0 + M\delta_i, \quad \text{for all } i = 1, \ldots, Q \tag{89}$$

$$w^T x^i \ge w_0 + \epsilon - M(1 - \delta_i), \quad \text{for all } i = 1, \ldots, Q \tag{90}$$

$$\delta_i \in \{0, 1\}, \quad \text{for all } i = 1, \ldots, Q \tag{91}$$

$$-1 \le \alpha_j \le 1, \quad \text{for all } j = 0, \ldots, n \tag{92}$$

where $X =: \{x^1, \ldots, x^Q\}$, and M and ϵ are a large and a small positive value, respectively.

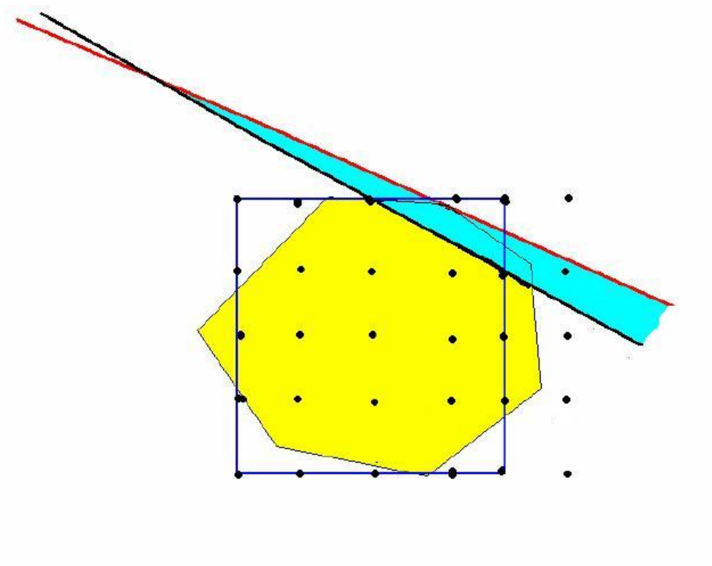
Notice that $u, w, w_0, \alpha, \alpha_0, \delta$ are all variables.

- The idea of the model above is to certify the validity of $w^T x \le w_0$ for P (where w and w_0 are now variables) by using Farkas' characterization (88).

  Because of (89), a point $x^i \in X$ can violated the inequality $\alpha^T x \le \alpha_0$ only by setting $\delta_i = 1$

  in which case (90) imposes that the valid inequality $w^T x \le w_0$ cuts it off (hence this point cannot be feasible for the original ILP model).

A geometrical interpretation



$$\max \quad \alpha^T x^* - \alpha_0 \tag{93}$$

$$w^T \le u^T A,\; w_0 \ge u^T b,\; u \ge 0 \tag{94}$$

$$\alpha^T x^i \le \alpha_0 + M\delta_i, \quad \text{for all } i = 1, \ldots, Q \tag{95}$$

$$w^T x^i \ge w_0 + \epsilon - M(1 - \delta_i), \quad \text{for all } i = 1, \ldots, Q \tag{96}$$

$$\delta_i \in \{0, 1\}, \quad \text{for all } i = 1, \ldots, Q \tag{97}$$

$$-1 \le \alpha_j \le 1, \quad \text{for all } j = 0, \ldots, n \tag{98}$$

- The solution of the MIP separation model can be obtained along the same lines as for its LP counterpart:

  Find an optimal solution $(u^*, w^*, w_0^*, \alpha^*, \alpha_0^*, \delta^*)$ of a **restricted** MIP separtion problem taking into account only a subset of points $x^1 \cdots x^h$.

  Invoke the KP oracle to solve

$$\max\{\alpha^* y : y \in KP(w^*, w_0^*)\}$$

  so as to certify the validity of $\alpha^* x \leq \alpha_0^*$ for the current KP relaxation $KP(w^*, w_0^*)$...

  ... or else to produce a new point x^{h+1} to be inserted in the MIP separation model (along with the corresponding variable δ_{h+1}), and repeat.