An Hybrid branch-and-cut for solving MINLPs

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CMU-IBM Collaboration

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- Develop algorithms for MINLP.
- Implement and release as open source software (COIN-OR).
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Participants

Carnegie Mellon

- Larry Biegler
- Pierre Bonami
- Gerard Cornuéjols
- Ignacio E. Grossmann
- Carl D. Laird
- François Margot
- Nick Sawaya

IBM

- Andrew R. Conn
- Laszlo Ladanyi
- Jon Lee
- Andrea Lodi
- Andreas Waechter

Solver for Mixed Integer Nonlinear Programming:

$$(MINLP) \begin{cases} \min f(x) \\ \text{s.t.} \\ g(x) \leq 0, \\ x \in X, \ x_i \in \mathbb{Z} \forall i \in \mathcal{I}. \end{cases}$$

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- $f: \ X \ \rightarrow \mathbb{R}$,
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f and g convex:

Continuous relaxation "easily" solvable, three exact algorithms :

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2. Outer Approximation decomposition (Duran and Grossmann 1986),
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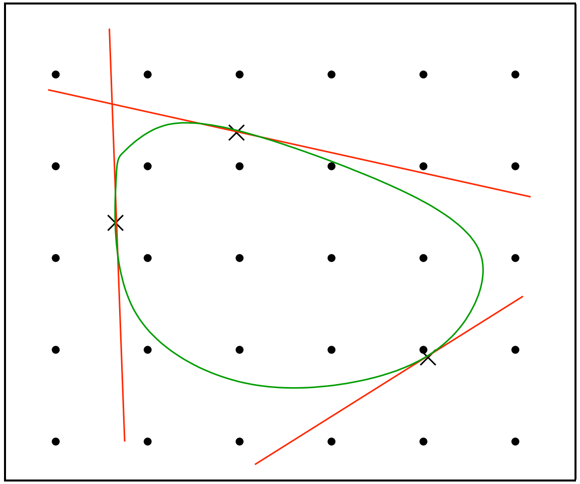
Non-convex MINLP

f or g non-convex:
Only compute local optima of continuous relaxations.
Uses NLP based branch-and-bound as an heuristic for searching good solutions.

$$\min f(x)$$
s.t.
$$g(x) \leq 0,$$
$$x \in X,\ x_i \in \mathbb{Z}\ \forall i \in \mathcal{I}.$$

(assume linear objective)



Idea: linearize constraints at different points and build an equivalent MILP:

$$(OA) \begin{cases} \min f(x) \\ J_g(x^k)^T \left(x - x^k\right) + g(x^k) \leq 0 \\ \qquad\qquad \forall (x^k) \in \mathcal{T} \\ x \in X,\ x_i \in \mathbb{Z}\ \forall i \in \mathcal{I}. \end{cases}$$

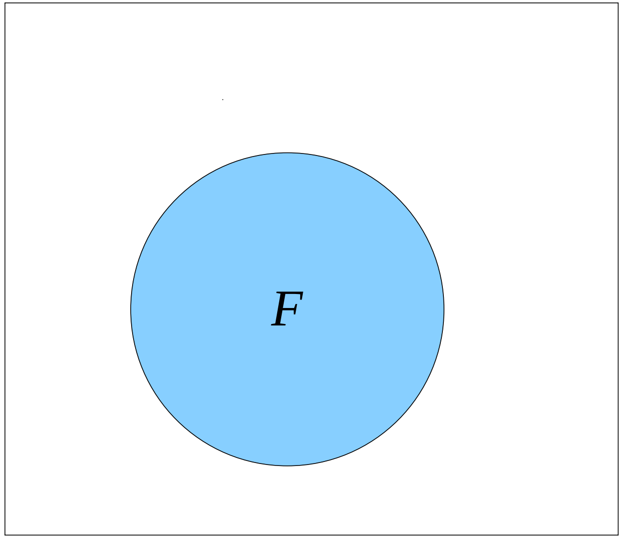
$\mathcal{T}$ contains suitably chosen linearization points.

Outer approximation constraints

Let $F := \{x : x \in X : g_i(x) \leq 0\}$
($g_i : \mathbb{R}^n \to \mathbb{R}$ convex.)
Outer approximation constraint in $\bar{x}$:

$$\nabla g_j(\bar{x})^T (x - \bar{x}) + g_j(\bar{x}) \leq g_j(x) \leq 0.$$

(valid for F by convexity of g_j and definition of F.)



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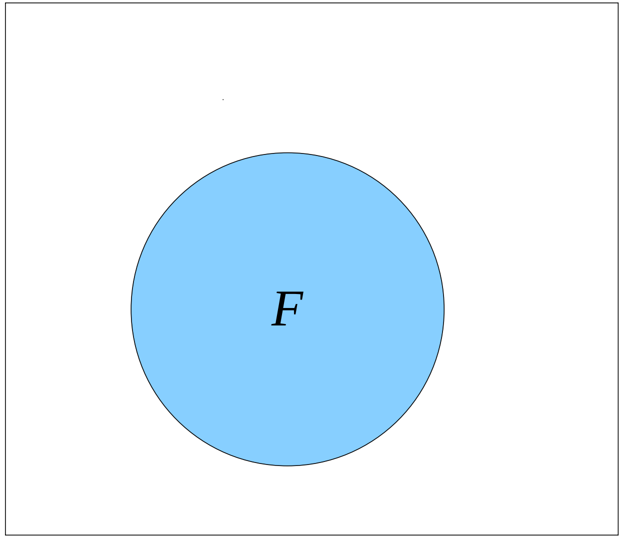

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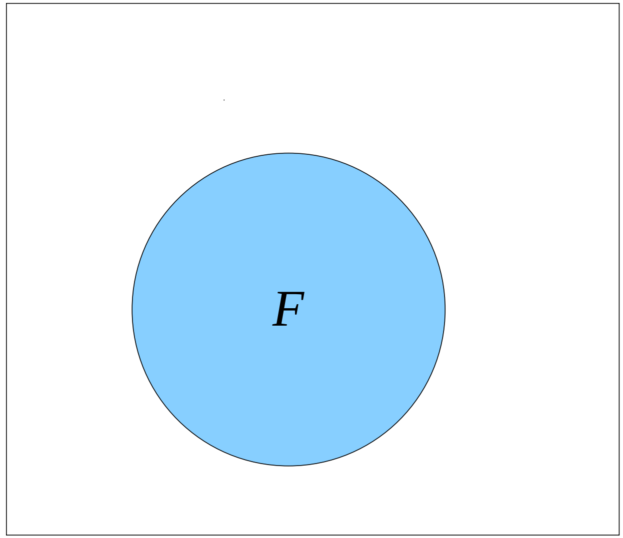

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- If $g(\bar{x}) < 0$ non-tight constraint.
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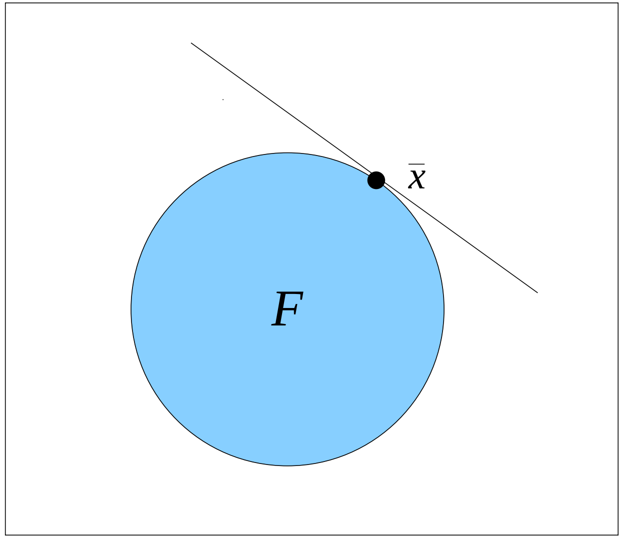

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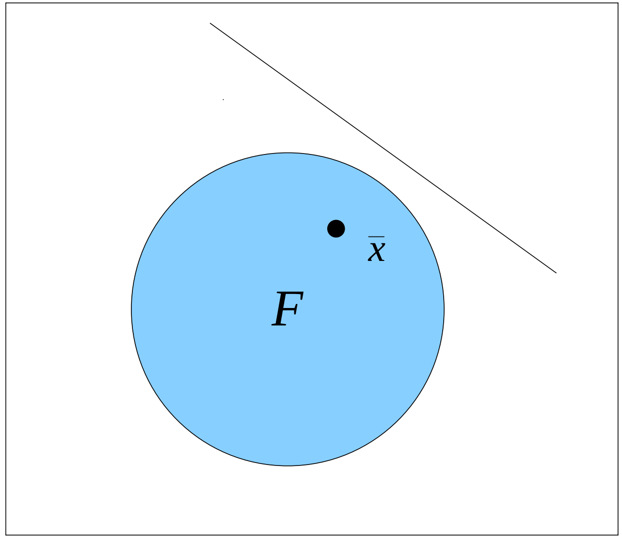

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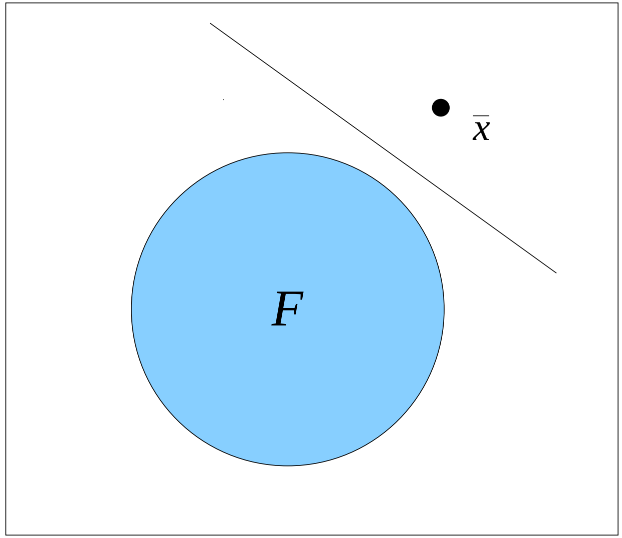

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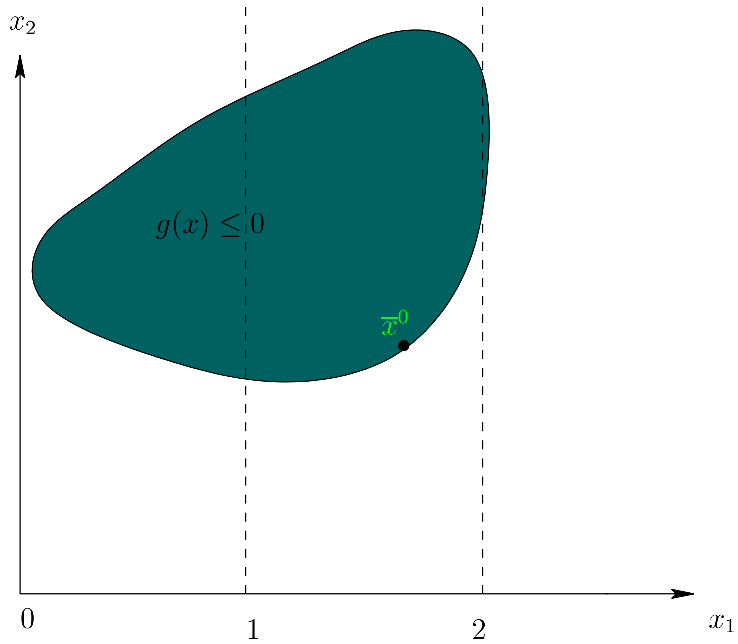

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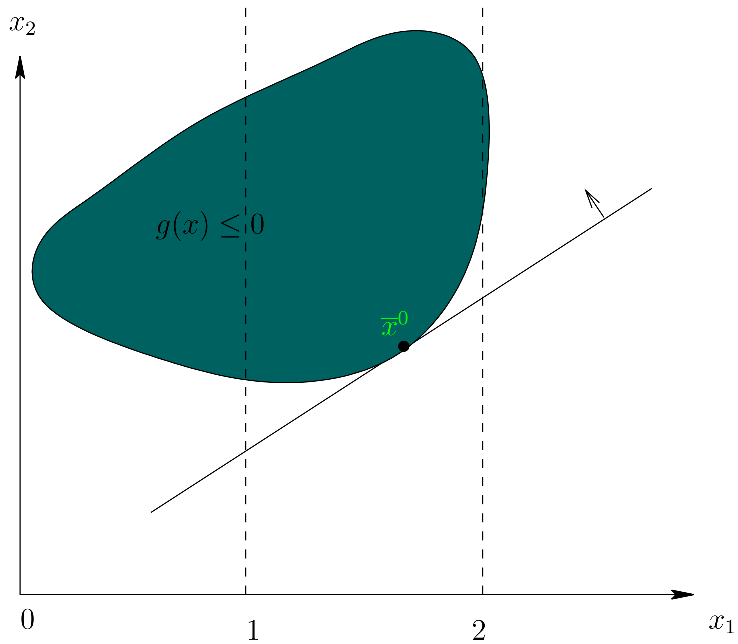

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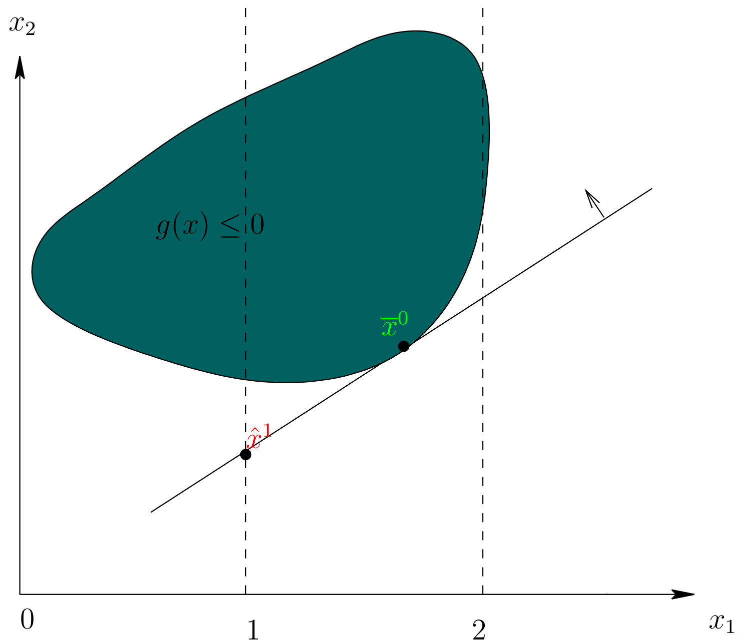

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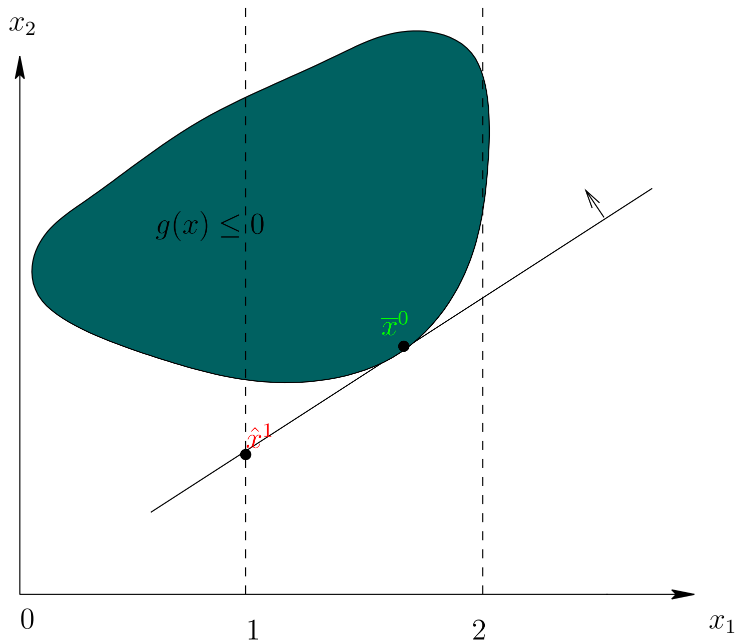

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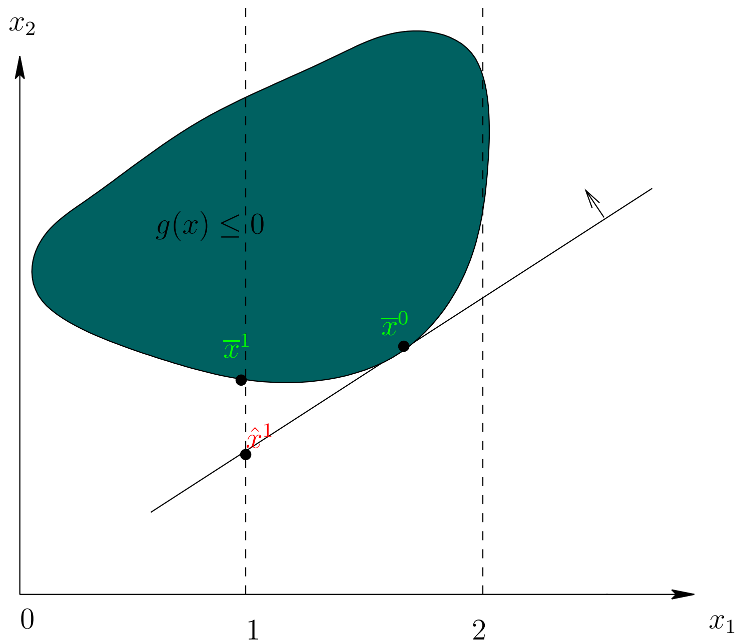

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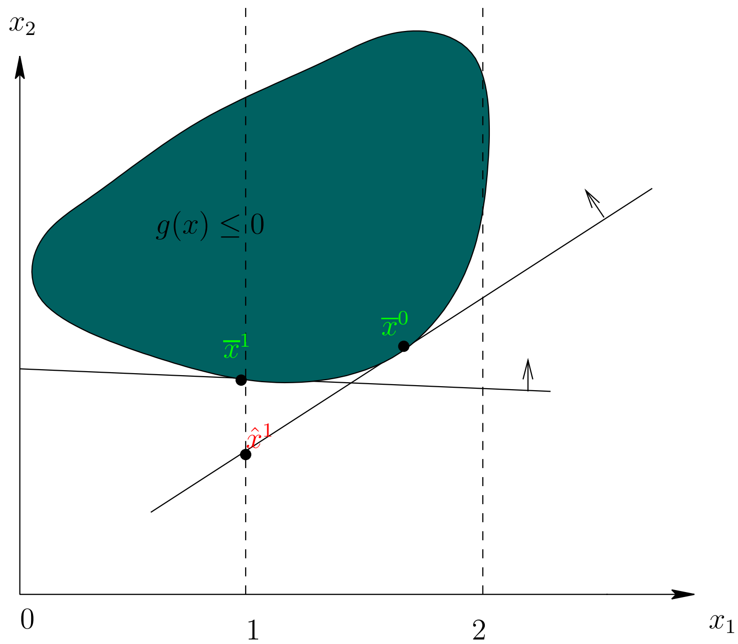

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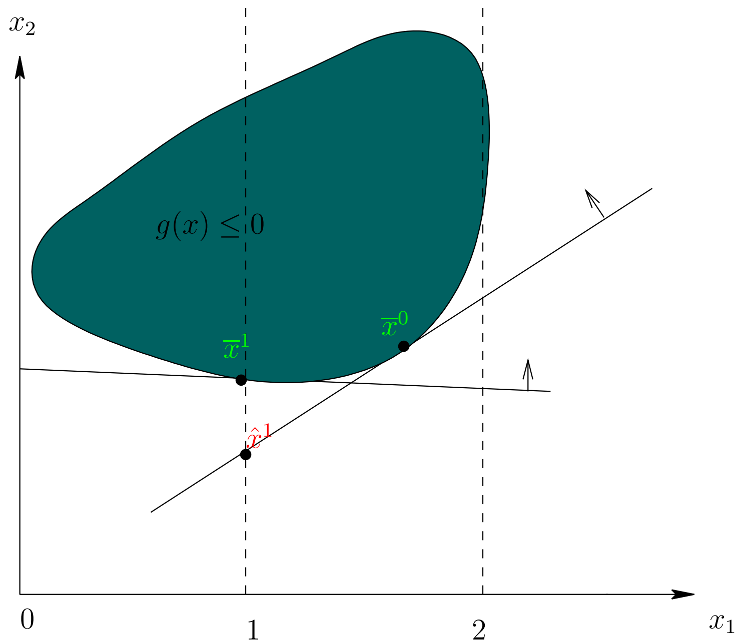

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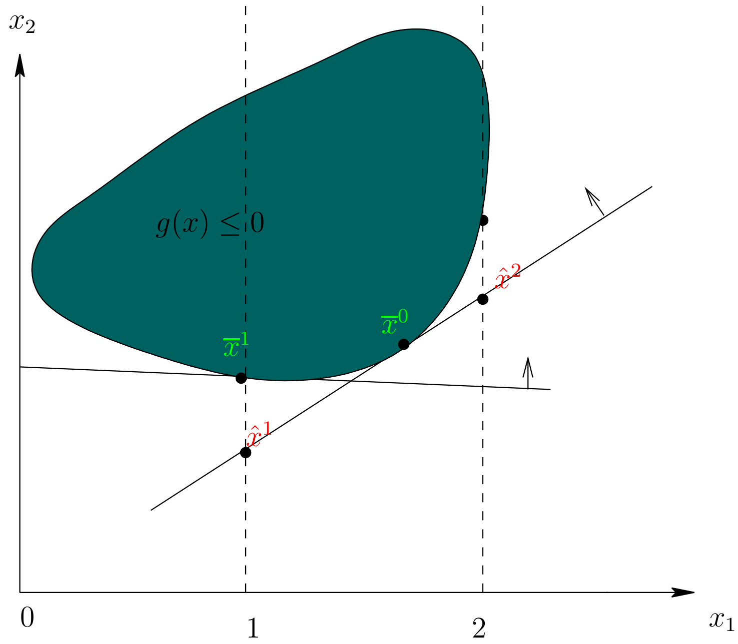

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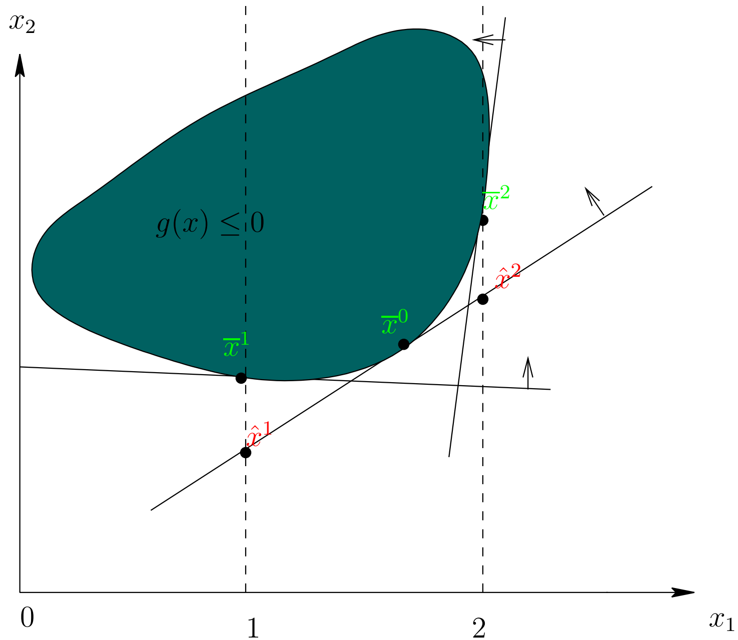

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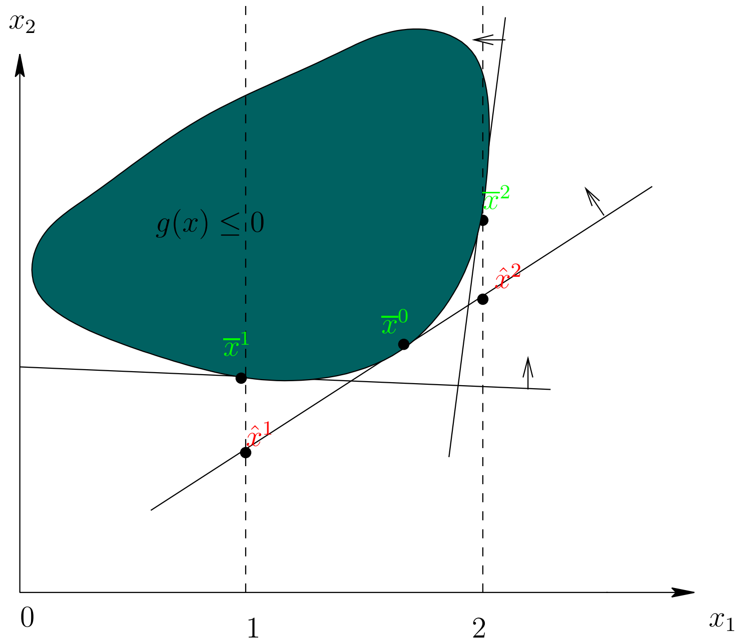
- Add $\bar{x}^1$ to $\mathcal{T}$ and iterate.
- Until either MILP is infeasible or the lower bound is equal to the upper bound .



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OA decomposition properties

If constraints qualification holds at every optimum of NLP solved:

- Solve to optimality problems defined by convex constraints.
- Finite termination if x bounded.

Has to solve a sequence of MINLP's ($> 95\%$ of computing time in our experiments).

Alternative approach [Quesada, Grossmann, 92]

- Perform a single branch-and-cut.
- Alternate between solving NLPs and LPs.
- NLP solved to find feasible solutions and improve outer approximation
- Use LP to obtain lower bounds and solutions to branch on

- start by solving continuous relaxation to get initial outer approximation.

[Quesada, Grossmann, 92]

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LP/NLP branch-and-cut

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- Fathom nodes on bounds and infeasibility only.

[Quesada, Grossmann, 92]

Disadvantages of LP/NLP branch-and-bound

- At the top of the tree outer approximation only based on continuous relaxation.
- Approximation not improved until first integer feasible solution is found.
- At that point tree may already have a large number of nodes.

Improvements

Solve more NLP's

- Initialize algorithm with a few iterations of OA decomposition.
  - Helps in finding feasible solution early.
  - Sometimes sufficient to solve problem.
- Solve NLP relaxation every l nodes:
  - LP relaxation at node is then equal to NLP relaxation.

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$$(OA)_F(\mathcal{T}) \begin{cases} \min f(x,y) \\ J_g(\bar{x})^T (x - \bar{x}) + g(\bar{x}) \leq 0 \quad \forall \bar{x} \in \mathcal{T} \\ x \in X \cap F. \end{cases}$$

  (F is the modified feasibility set at current node)

  - if $(OA)_F(\mathcal{T})$ is integer feasible solve NLP:

$$\begin{cases} \min f(x) \\ g(x) \leq 0, \\ x \in X, \ x_i = \hat{x}_i \ \forall i \in \mathcal{I} \end{cases}$$

  - add its solution $\bar{x}$ to $\mathcal{T}$, and repeat while solution to $(OA)_F(\mathcal{T})$ is integer feasible.
- Fathom nodes on bounds and infeasibility only.

- start by performing t sec. of outer approximation decomposition.
- At each node of the tree search
  - Every l nodes solve NLP relaxation:

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  - solve linear outer approximation at current node:

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  - Strengthen outer approximation with MILP cutting planes methods
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- Written in C++.
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Building blocks

Components from COIN-OR (www.coin-or.org):

- branch-and-bound, branch-and-cut framework: Cbc,
- NLP solver Ipopt,
- MILP solver Cbc (alternatively Cplex),
- LP solver Clp,
- Cutting plane generation Cgl (generators for OA constraints).

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Availability

To be released soon under Common Public License.

I-BB (NLP only based branch-and-bound)

- branch-and-bound framework based on `Cbc`,
- nodes solved by `Ipopt` with enhanced warm-starting capabilities.
- Special features for non-convex problems :
  - Solve each nodes with several randomly chosen starting points,
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Standard OA decomposition

- Implemented as a cut generator to be used in other algorithms.
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I-Hyb (Hybrid)

Quesada-Grossmann improved with:

- Initialization with a short time of $I - OA$.
- Solve NLP's every l nodes.
- Incorporation of MILP techniques:
  - Cgl cut generators,
  - Strong branching, Reliability branching

A Library of convex MINLPs

About 150 MINLP's from different sources

- Existing problems from the literature : layout and trimloss problems (T. Westerlund et al.)
- Water network problems (C.D. Laird)
- Disjunctive problems formulated both with big-M and convex-hull formulation (N. Sawaya)

available in GAMS .gms and AMPL .nl formats at:
http://egon.cheme.cmu.edu/ibm/page.htm

Computational experiments

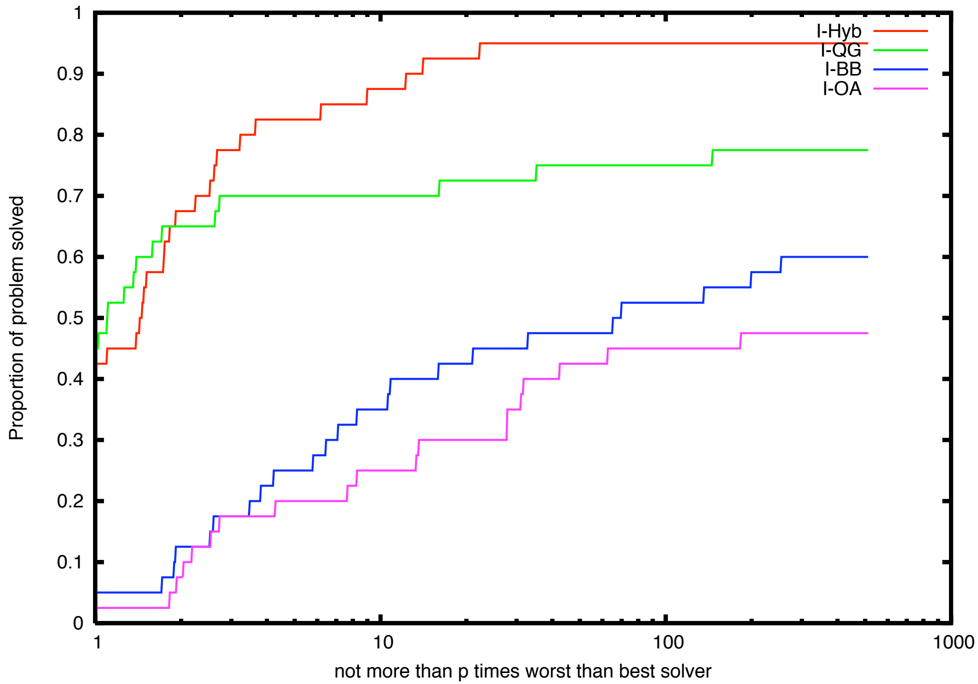
Comparisons on 38 problems from the library:

1. Comparison of I-BB, I-OA, I-QG and I-Hyb,
2. Comparison of I-BB I-OA and I-Hyb with two commercial solvers:
    - SBB: Nonlinear branch-and-bound based on CONOPT.
    - DICOPT: OA decomposition based on CONOPT/CPLEX.

Settings for I-Hyb

- Perform 30 seconds of I-OA at the root node.
- Solve NLP relaxation every 10 nodes.
- Strong branching on LP relaxation and pseudo-costs.
- Uses MIG, MIR and Covers.

Comparison of I-BB, I-OA, I-QG, I-Hyb



Runs on an Optetron cluster.
Time limit of 3 hours.

Performance plot

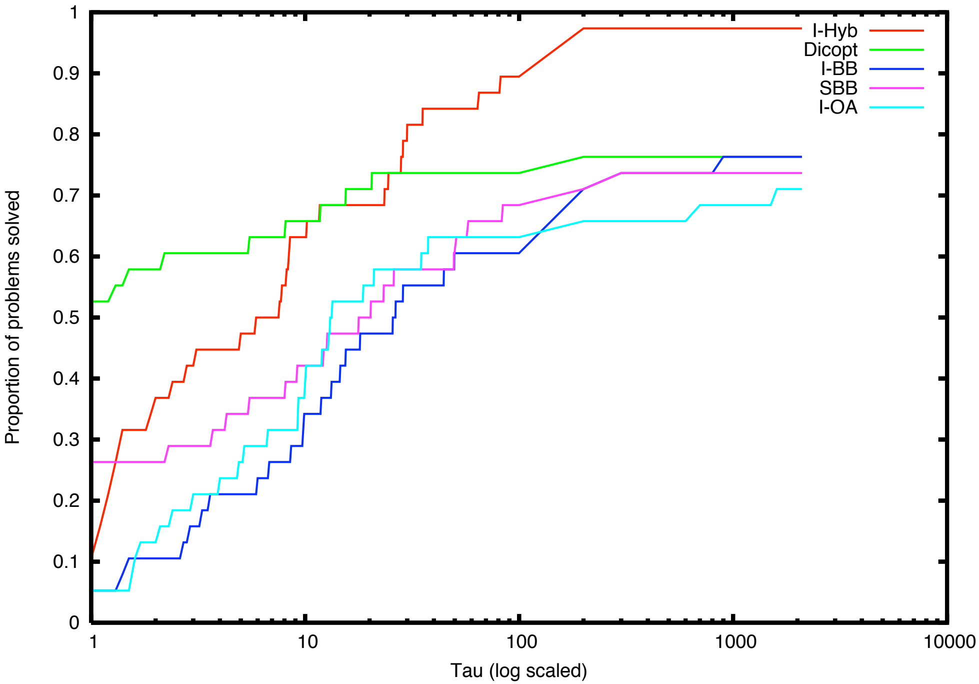
For each value of p and algorithm A gives the proportion of problem solved by A in

$$p \times T_{\min} \text{ seconds.}$$

(Where $T_{\min}$ is the running time of the best algorithm.)

- For $p = 1$: proportion of problems where A is the fastest algorithm
- For $p = \infty$: proportion of problems solved by A in time limit.

BONMIN's I-BB, I-OA, I-Hyb, Dicopt and Sbb



- Dicopt solves 20 of the 38 problems the fastest (≤ 3 minutes)
- I-Hyb solves the most problem when given 45 more seconds of computing time than Dicopt

Comparison of our branch-and-bound and Sbb

- I-BB is slightly slower but compares well with Sbb (on average 760 sec. vs 615 sec. on problems solved optimally by both)
- Number of nodes are comparable
- I-BB is slightly faster per node on our test set.

Comparison of I-OA with Dicopt

- I-OA is significantly slower than Dicopt.
- Takes less iterations but MILPs are much slower to solve (uses Cbc vs. Cplex).

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How: Do an OA decomposition oriented towards integer feasibility.

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Feasibility Pump for MILP [Fischetti, Glover, Lodi 2004]

$$\begin{cases} Ax \leq b \\ x \in \mathbb{Z}^n \end{cases}$$

Construct two sequences of points:

- $\bar{x}^1, \ldots, \bar{x}^k$ satisfying $Ax \leq b$, by solving LPs.
- $\hat{x}^1, \ldots, \hat{x}^k$ satisfying $x \in \mathbb{Z}^n$ by rounding.

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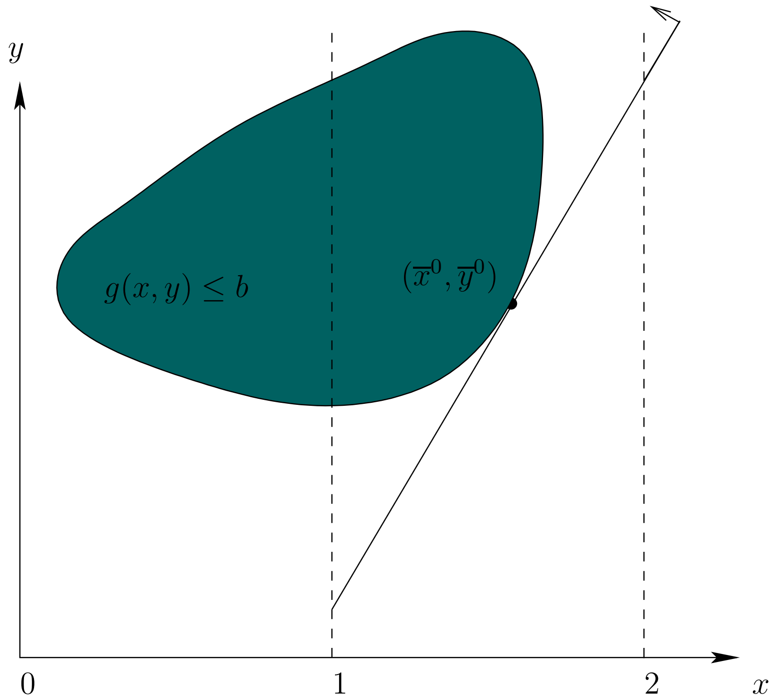
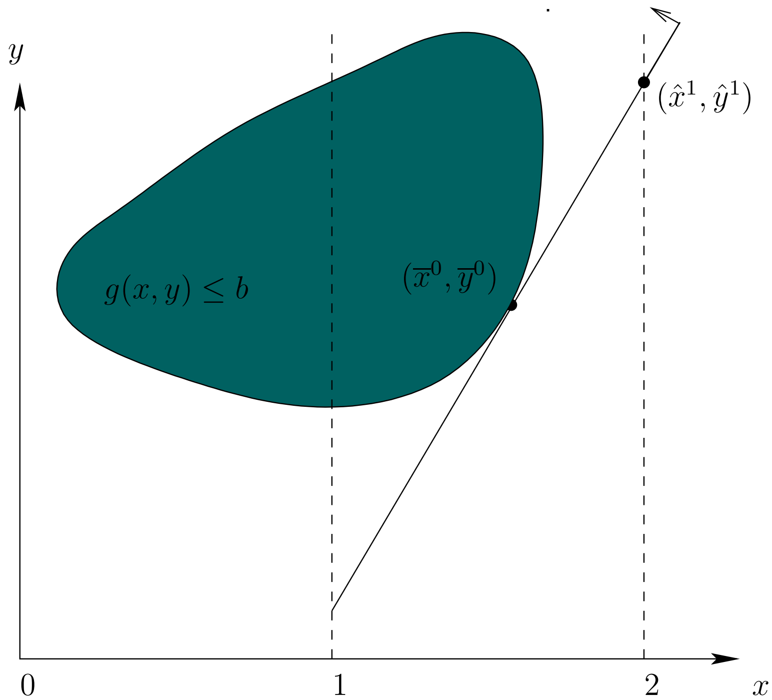
- Build an outer approximation of feasibility region.



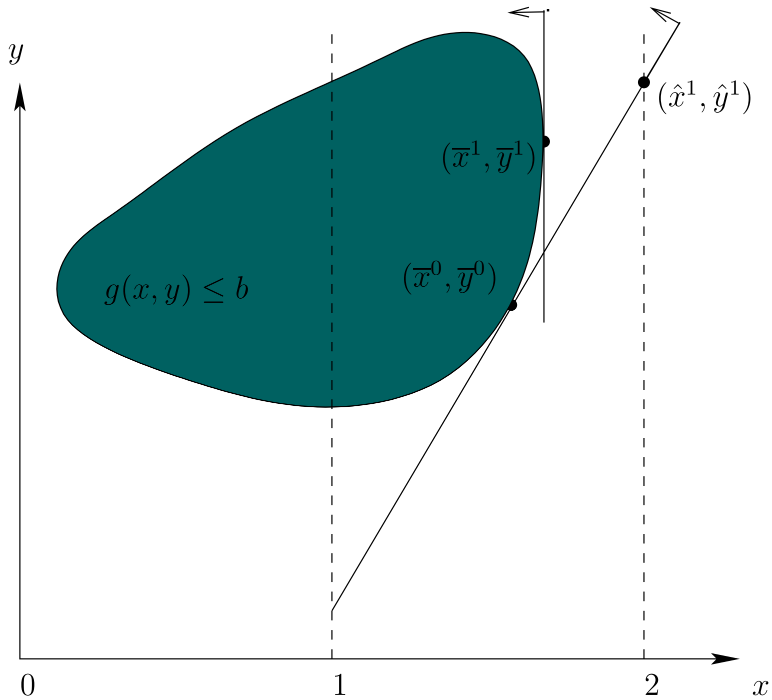
- Start with any solution of continuous relaxation $(\bar{x}^0, \bar{y}^0)$.



- Start with any solution of continuous relaxation $(\bar{x}^0, \bar{y}^0)$.
- $\mathcal{T} = (\bar{x}^0, \bar{y}^0)$.

- Find point minimizing $||x - \bar{x}^0||_1$ in current outer approximation:

$$(FOA)^1 \begin{cases} \min ||x - \bar{x}^0|| \\ g(\overline{x}^k, \overline{y}^k) + J_g(\overline{x}^k, \overline{y}^k)+)^T \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \overline{x}^k \\ \overline{y}^k \end{pmatrix} \right) \leq 0 \\ x \in \mathbb{Z}^{n_1},\ y \in \mathbb{R}^{n_2} \end{cases}$$



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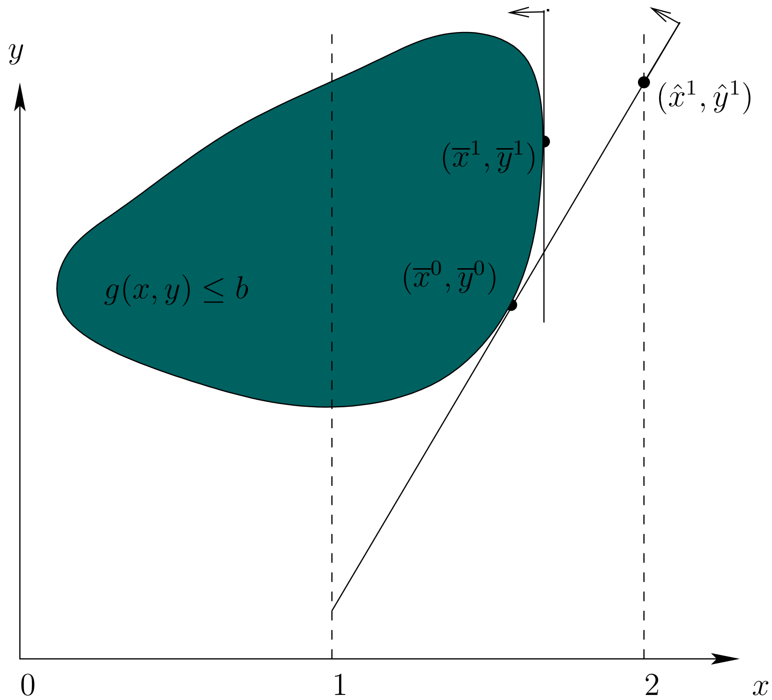

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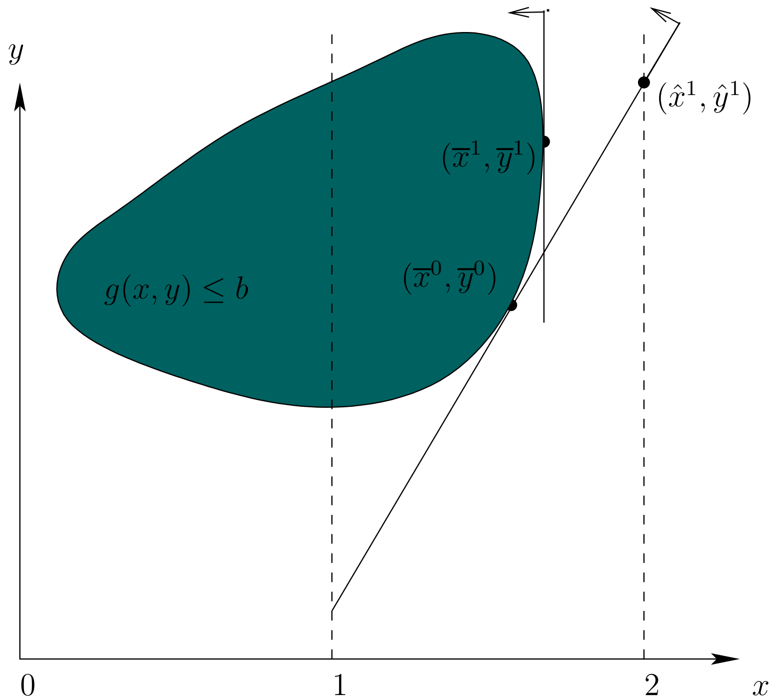

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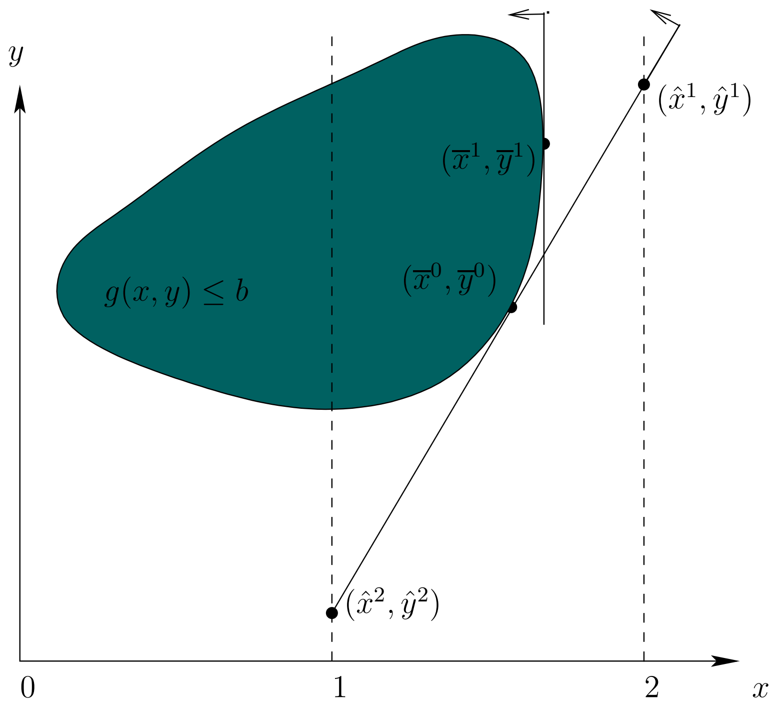

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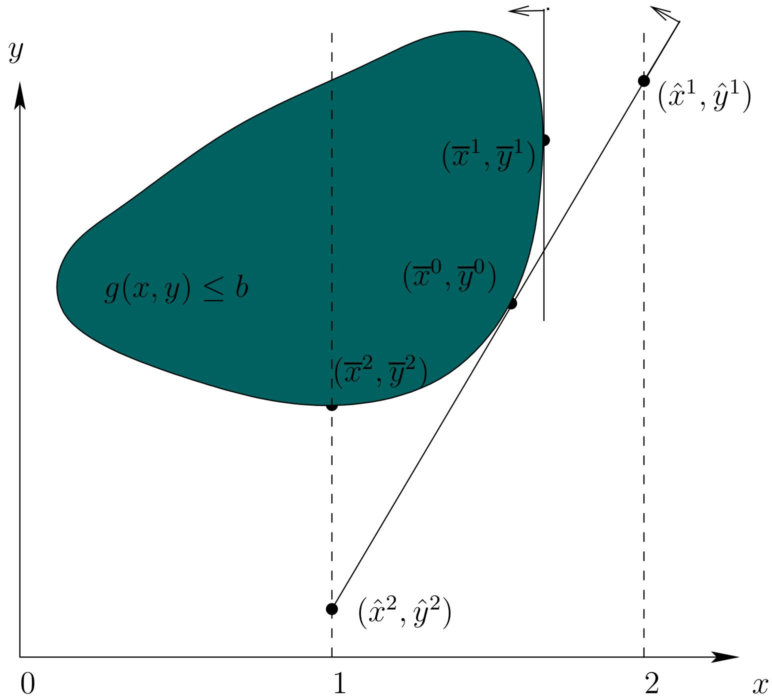

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Termination

FP can not cycle (if x is bounded: finite termination):

- If all functions g_i are convex and constraint qualification holds at every NLP optimum.
- If constraint qualification does not hold at every NLP add cut:

$$(\bar{x}^i - \hat{x}^i)^T (x - \bar{x}^i) \geq 0$$

- If functions g_i are not convex but the region $\{g(x, y) \leq 0\}$ is add only binding OA constraints at $\bar{x}^i$.
- MILPs don't have to be solved to optimality.

Iterated feasibility pump (IFP)

After a feasible solution of cost $\alpha = f(\bar{x}, \bar{y})$ has been found.
Add the constraint

$$f(x, y) \leq \alpha - \epsilon$$

to problem formulation and relaunch FP.

Implementation

- Implemented as a stand-alone heuristic.
- Ipopt3.0 for solving the NLPs.
- Cplex9.0 for solving MILPs.

Test problems

65 convex MINLPs

- 12 from literature
- 43 from our library

Comparison with classical OA

- First feasible solution obtained by FP and OA.
- Best feasible solution obtained by IFP and OA after 1 minute.

First feasible solution

Time limit 2 hours.

- Quality of solution obtained by OA better than the one obtained by FP.
- FP much faster than OA (4 problems take more than 10 sec. with FP, 21 with OA)
- FP finds a feasible solution to all 65 problems, OA does not for 5 problems.
- trimloss6-7-12 no feasible solution known before.

1 minute of IFP *vs.* OA

- IFP finds solution for 63 problems, OA for 50.
- Quality of solutions very comparable.
- IFP proves optimality of 30 problems, OA of 38.

Combination of OA and FP.

Principle

- Start by performing one minute of IFP to get a good feasible solution (and OA constraints).
- Launch a classical OA decomposition but every time the NLP is infeasible launch an FP to try to obtain a feasible solution.

- Solve the continuous relaxation of $(MINLP)$:

$$\begin{cases} \min f(x) \\ g(x) \leq 0, \\ x \in X, \end{cases}$$

- Construct MILP with linearization in $\bar{x}^0$ ($\mathcal{T} = \{\bar{x}^0\}$) :

$$\begin{aligned} &\min f(x) \\ &J_g(\bar{x}^0)\left(x - \bar{x}^0\right) + g(\bar{x}^0) \leq 0 \\ &x \in X,\ x_i \in \mathbb{Z}\ \forall i \in \mathcal{I}. \end{aligned}$$

  Solution $\hat{x}^1$ gives a lower bound on $(MINLP)$.
- From the solution $\hat{x}^1$ build an NLP with integer variables fixed:

$$\begin{cases} \min f(x) \\ g(x) \leq 0, \\ x \in X,\ x_i = \hat{x}_i^1\ \forall i \in \mathcal{I} \end{cases}$$

  - If feasible the solution $\bar{x}^1$ gives upper bound .
  - Otherwise, $\bar{x}^1$ minimizes constraints infeasibility. Linearization cuts off $\{x \in X : x = \hat{x}_i^1\}$
- Add $\bar{x}^1$ to $\mathcal{T}$ and iterate.
- Until either MILP is infeasible or the lower bound is equal to the upper bound .

- Solve the continuous relaxation of $(MINLP)$:

$$\begin{cases} \min f(x) \\ g(x) \leq 0, \\ x \in X, \end{cases}$$

- Perform 1 minute of IFP add all the NLP feasible points found to $\mathcal{T}$
- Construct MILP with linearization for all $\overline{x} \in \mathcal{T}$:

$$\begin{array}{l} \min f(x) \\ J_g(\bar{x})\,(x - \bar{x}) + g(\bar{x}) \leq 0 \quad \forall \bar{x} \in \mathcal{T} \\ x \in X,\ x_i \in \mathbb{Z}\ \forall i \in \mathcal{I}. \end{array}$$

  Solution $\hat{x}$ gives a lower bound on $(MINLP)$.
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  - If feasible the solution $\bar{x}$ gives upper bound .
  - Otherwise, Launch an FP for at most 2 minutes and 5 iterations add all NLP feasible solution to $\mathcal{T}$
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On a subset of 15 hardest problems with OA from previous experiment:

	OA enhanced by FP				OA			
Name	ub	time to find ub	lb	time to find lb	ub	time to find ub	lb	time to find lb
CLay0304M	40262.4	79	*	82	40262.4	12	*	14
CLay0305H	8092.5	4	*	32	8092.5	24	*	24
CLay0305M	8092.5	4	*	24	8092.5	75	*	75
fo7_2	17.74	4	*	103	17.74	20	*	128
fo7	20.72	260	*	260	20.72	24	*	197
fo8	22.38	573	*	835	22.38	727	*	906
fo9	23.46	1160	*	2613	23.46	5235	*	6024
o7_2	116.94	189	*	2312	118.86	5651	114.08	7200
o7	131.64	5	*	6055	none	—	122.79	7200
SLay10M	129580	1778	*	3421	129580	336	128531	7200
trimloss2	5.3	0.17	*	0.22	5.3	0.21	*	0.21
trimloss4	8.3	10	*	423	8.3	785	*	785
trimloss5	10.7	485	3.31	7200	none	—	5.9	7200
trimloss6	16.5	2040	3.5	7200	none	—	6.5	7200
trimloss7	27.5	387	2.6	7200	none	—	3.3	7200
trimloss12	none	—	5.47	7200	none	—	9.58	7200

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trimloss6	16.5	2040	3.5	7200	none	—	6.5	7200
trimloss7	27.5	387	2.6	7200	none	—	3.3	7200
trimloss12	none	—	5.47	7200	none	—	9.58	7200

On a subset of 15 hardest problems with OA from previous experiment:

	OA enhanced by FP				OA			
Name	ub	time to find ub	lb	time to find lb	ub	time to find ub	lb	time to find lb
CLay0304M	40262.4	79	*	82	40262.4	12	*	14
CLay0305H	8092.5	4	*	32	8092.5	24	*	24
CLay0305M	8092.5	4	*	24	8092.5	75	*	75
fo7_2	17.74	4	*	103	17.74	20	*	128
fo7	20.72	260	*	260	20.72	24	*	197
fo8	22.38	573	*	835	22.38	727	*	906
fo9	23.46	1160	*	2613	23.46	5235	*	6024
o7_2	116.94	189	*	2312	118.86	5651	114.08	7200
o7	131.64	5	*	6055	none	—	122.79	7200
SLay10M	129580	1778	*	3421	129580	336	128531	7200
trimloss2	5.3	0.17	*	0.22	5.3	0.21	*	0.21
trimloss4	8.3	10	*	423	8.3	785	*	785
trimloss5	10.7	485	3.31	7200	none	—	5.9	7200
trimloss6	16.5	2040	3.5	7200	none	—	6.5	7200
trimloss7	27.5	387	2.6	7200	none	—	3.3	7200
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Parallel Implementation [L. Ladanyi]

- Using BCP as branch-and-cut framework,
- Prototypes of simplified I-Hyb and I-BB

Non-convex MINLPs

Trying to find heuristics to obtain good solutions in I-BB.

Stochastic programming (with M. Lejeune Tepper SoB)

Problems formulated as convex MINLPs

- Probabilistically constrained problems enforcing system/network reliability level
  - Reservoir management,
  - supply chain management,
  - financial applications (cash-matching)
- Robust/Probabilistic with random technology matrix problems integer constrained
  - integer constrained portfolio optimization problems.

IBM-CMU MINLP web site

http://egon.cheme.cmu.edu/ibm/page.htm

- Research reports :
  - An Algorithmic Framework for convex Mixed Integer Nonlinear Programs (with IBM-CMU group),
  - A Feasibility Pump for MINLP (with G. Cornujols, A. Lodi, F. Margot).
- Library of convex test problems available in Gams and Ampl .nl formats.